Why to love isogenies in 2024

Chloe Martindale

University of Bristol

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- Rich mathematical structure $\rightsquigarrow$ flexible post-quantum applications. Since 2018:
  - Only practical pq non-interactive key exchange
  - Fast, small key encapsulation mechanism
  - Two different signature schemes
  - Oblivious pseudorandom functions
  - Threshold schemes
  - ...

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  - Difficult to make concrete parameter choices.
- Slow: Orders of magnitude slower than ECC or the fastest pq option (structured lattices).

Example: CRS (Couveignes '97 Rostostev-Stolbunov '04)

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- S is a (specially chosen) set of elliptic curves
- H acts via isogenies (maps between elliptic curves)


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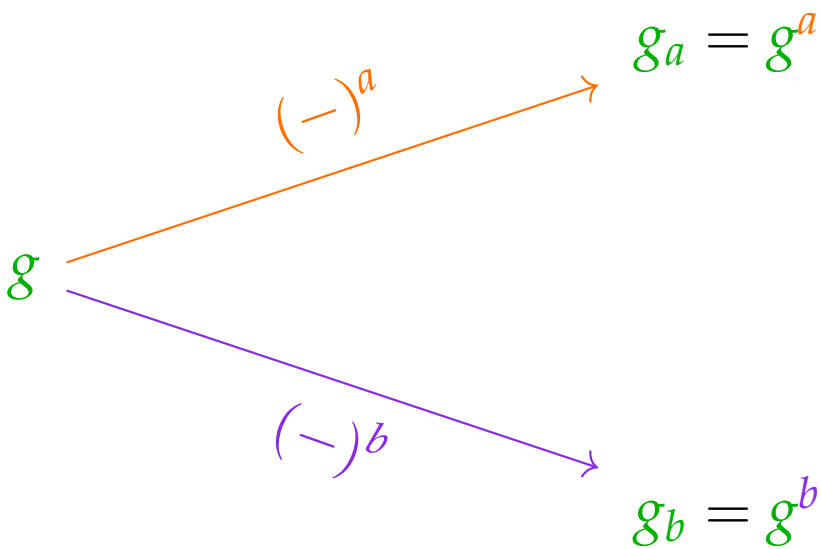
Evolution of key exchange

Diffie-Hellman



Colour code: Public, Alice's secret, Bob's secret

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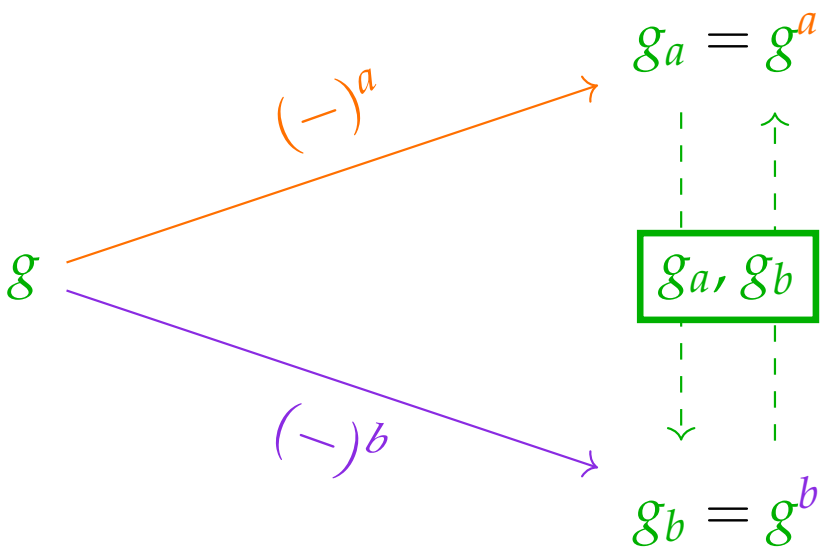

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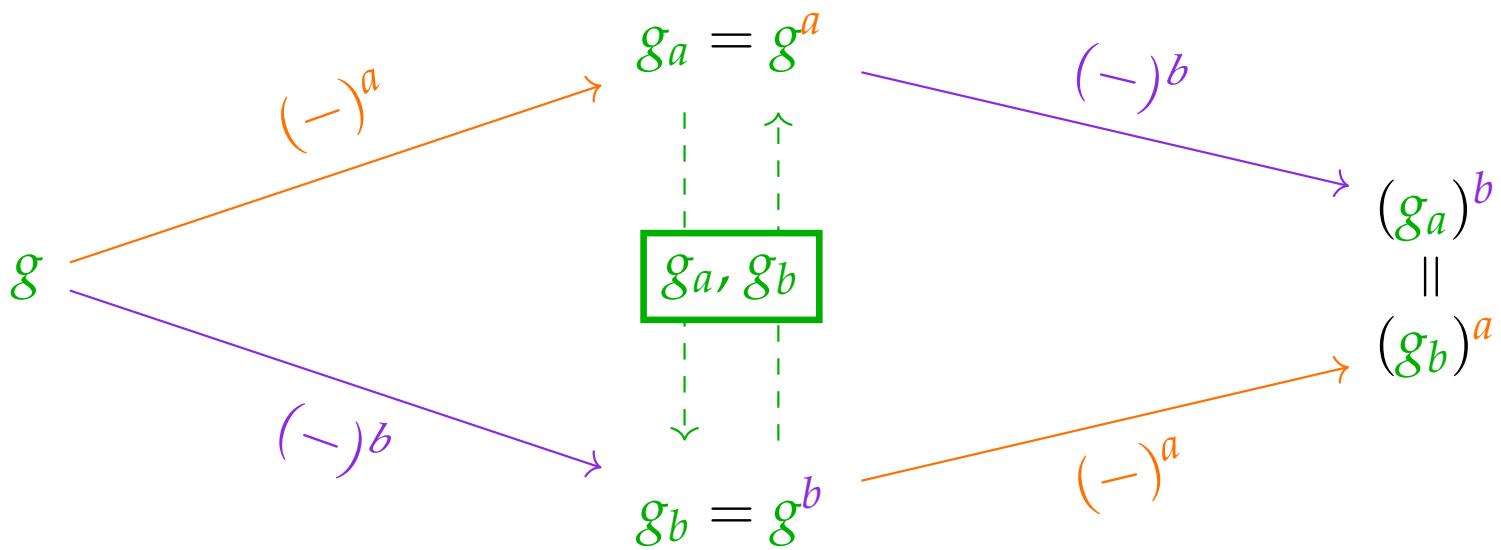

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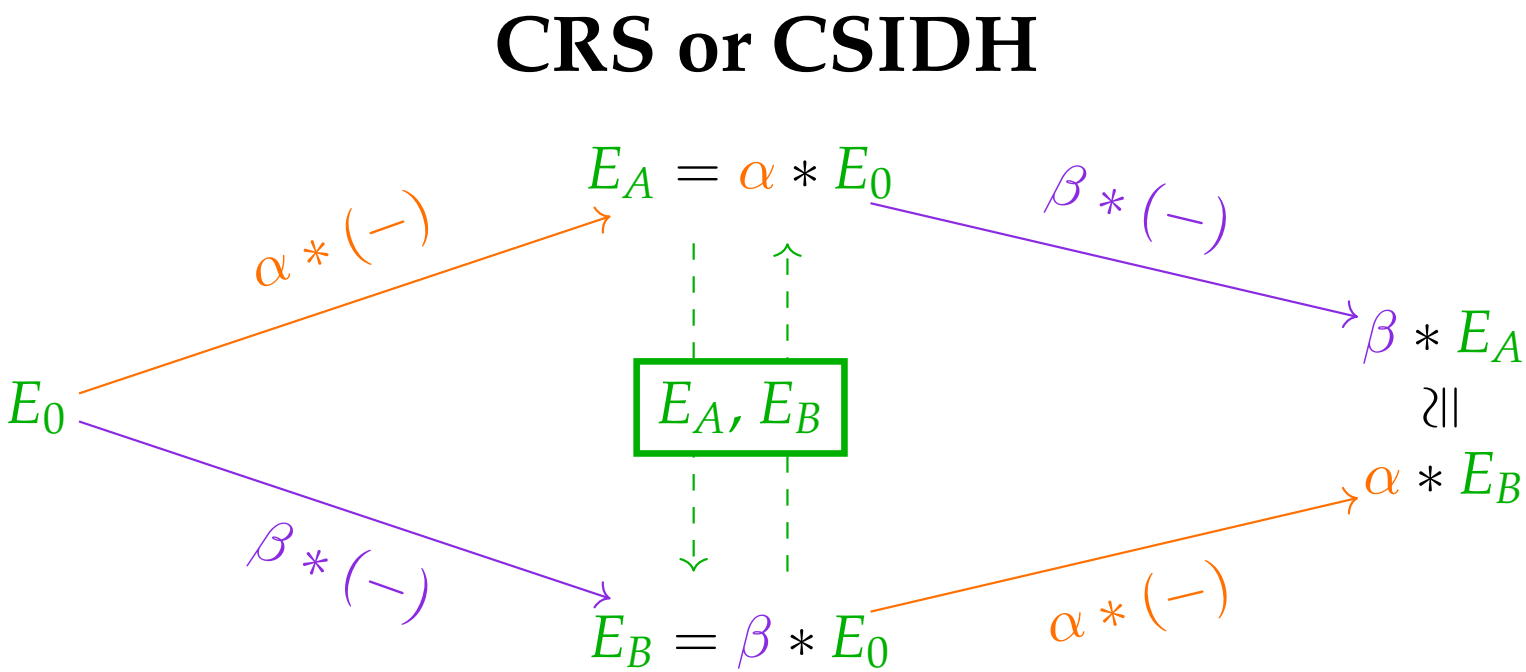

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Signatures (S '06, DG '18, BKV '19, DFKLMPW '23)

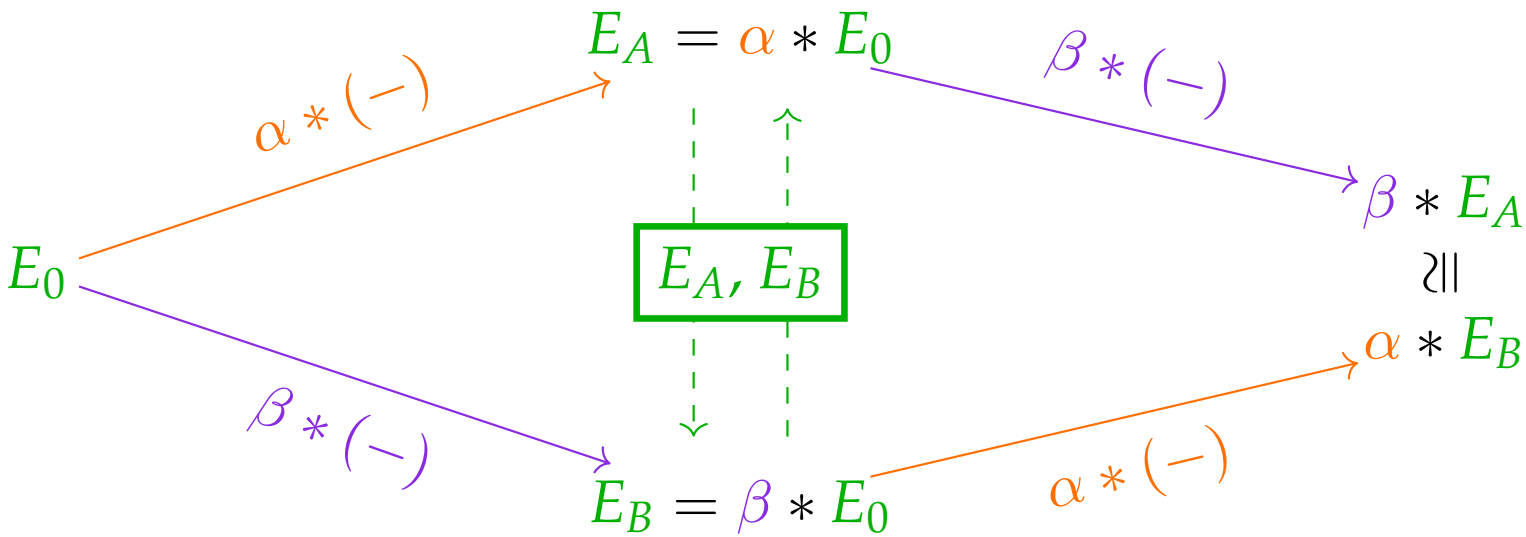
Identification scheme from $H \times S \to S$:

Prover	**Public**	**Verifier**
	$E \in S, \alpha_i \in H$	
$s_i \leftarrow \$\, \mathbb{Z}$		
$\mathsf{sk} = \prod \alpha_i{}^{s_i}$,		
$\mathsf{pk} = \mathsf{sk} * E$	$\xrightarrow{\mathsf{pk}}$	pk
	$\xleftarrow{c}$	$c \leftarrow \$\, \{0,1\}$
$t_i \leftarrow \$\, \mathbb{Z}$		
$\mathsf{esk} = \prod \alpha_i{}^{t_i}$,		
$\mathsf{epk}_1 = \mathsf{esk} * E$,		
$\mathsf{epk}_2 = \mathsf{esk} \cdot \mathsf{sk}^{-c}$	$\xrightarrow{\mathsf{pk},\mathsf{epk}_1,\mathsf{epk}_2}$	check:
		$\mathsf{epk}_1 = \mathsf{epk}_2 * ([\mathsf{sk}^c] * E)$.

After k challenges c, an imposter succeeds with prob 2^{-k}.

Evolution of key exchange

CRS or CSIDH



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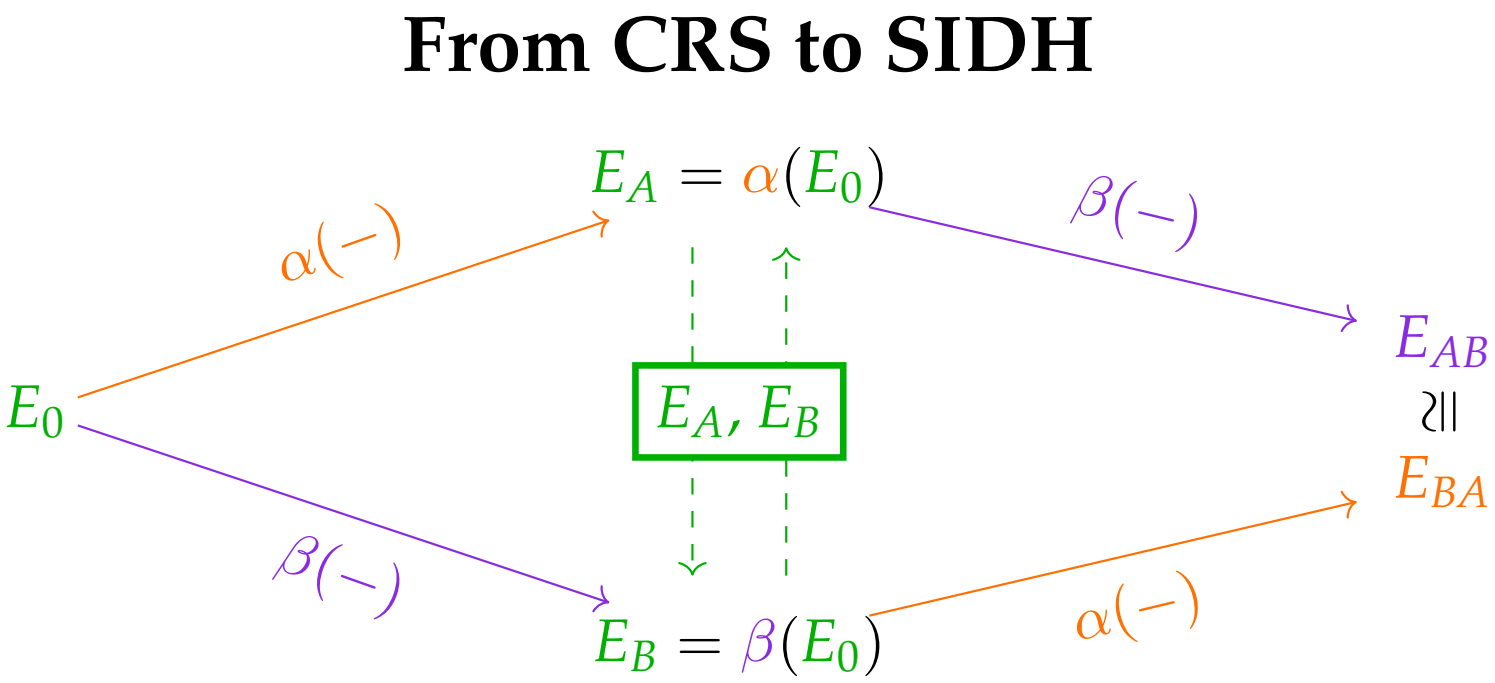

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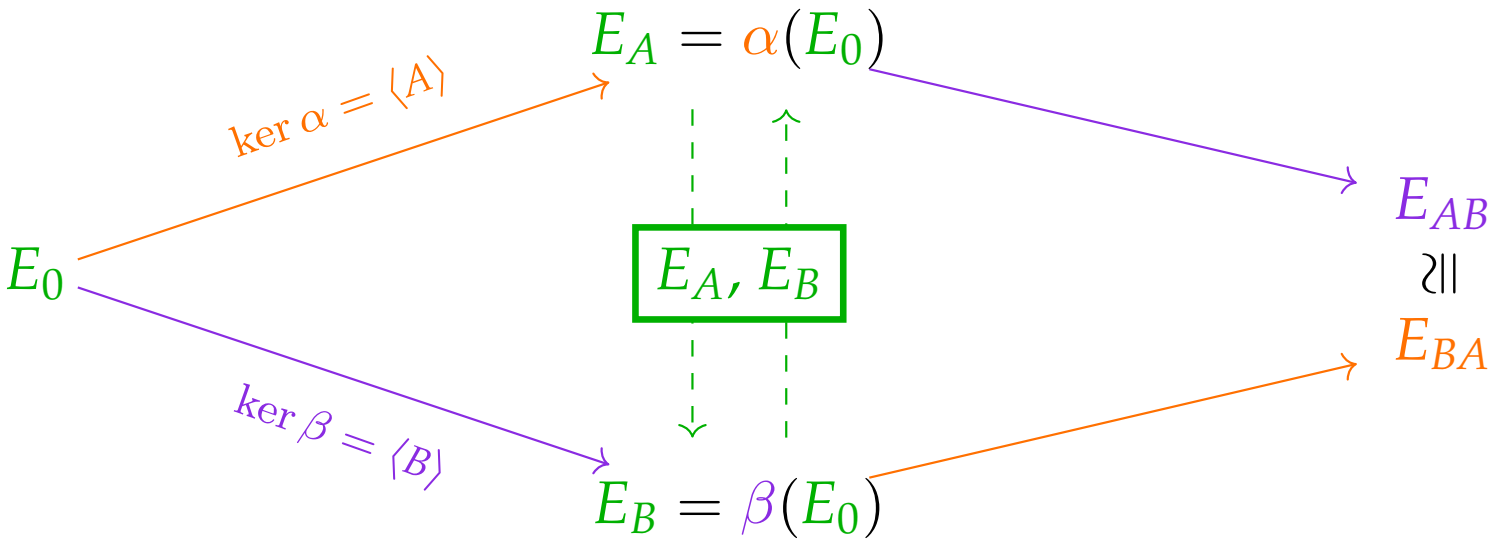

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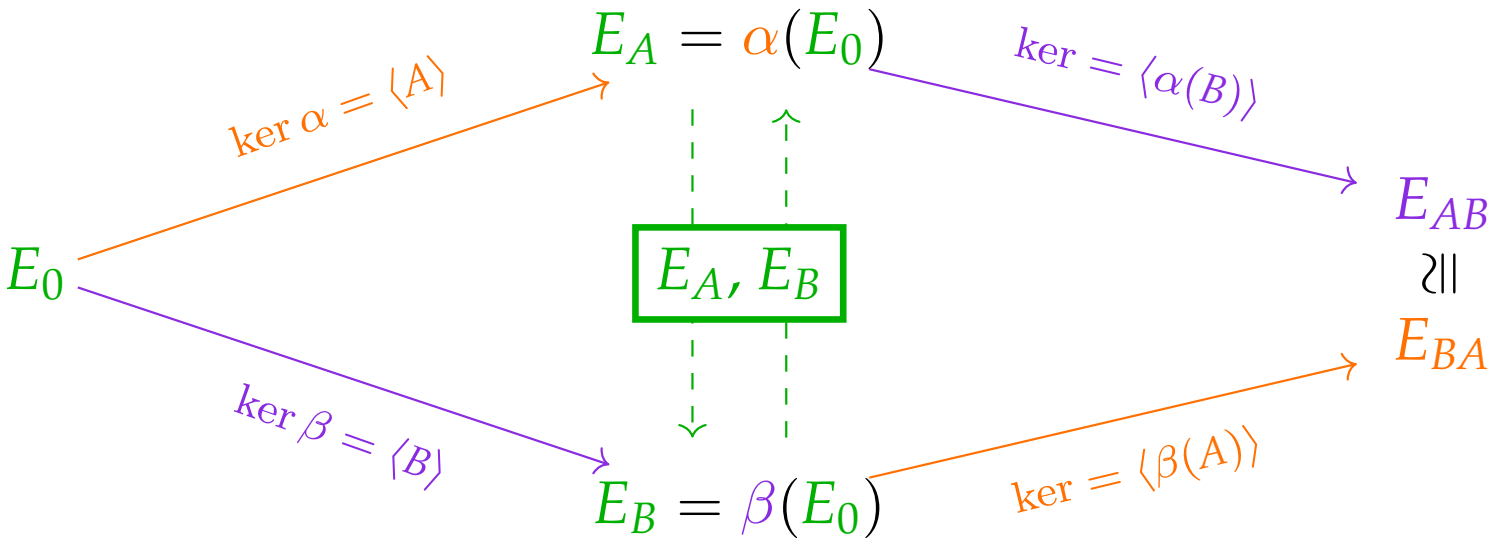

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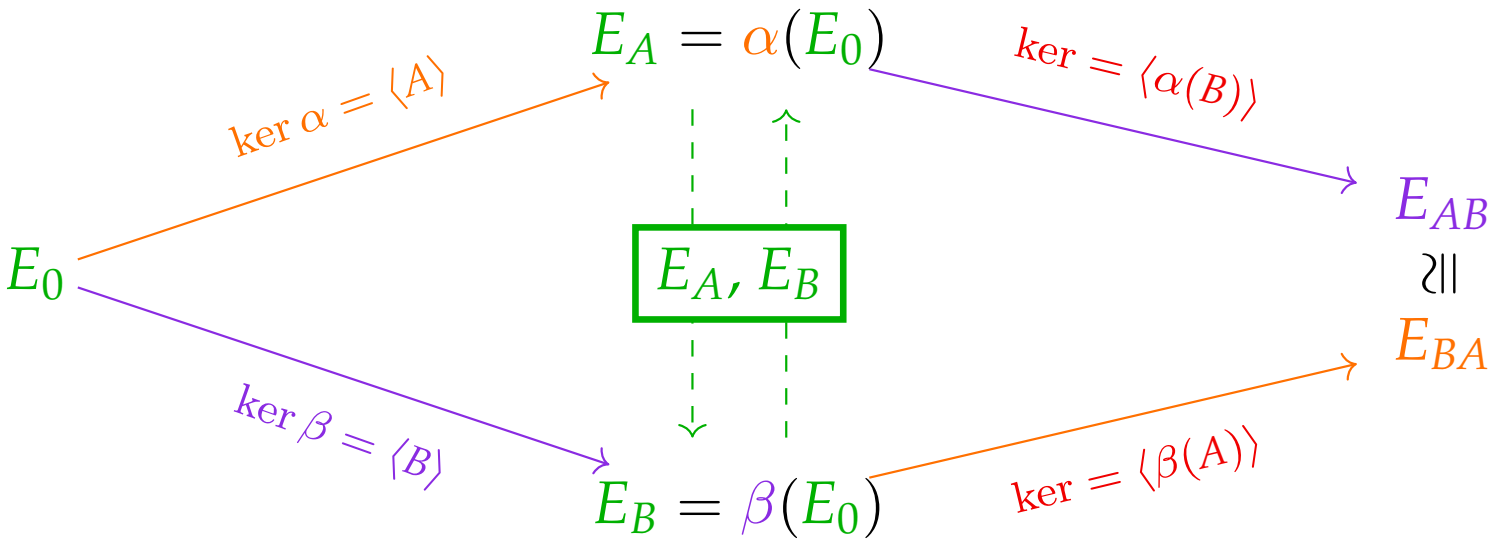

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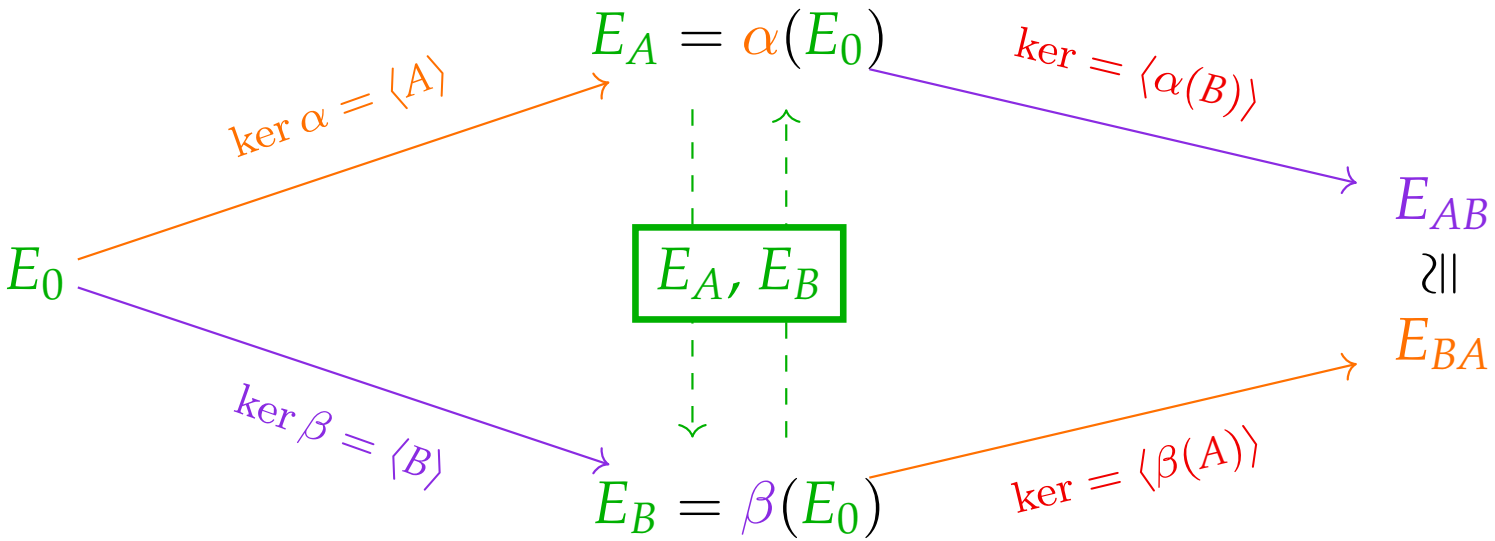

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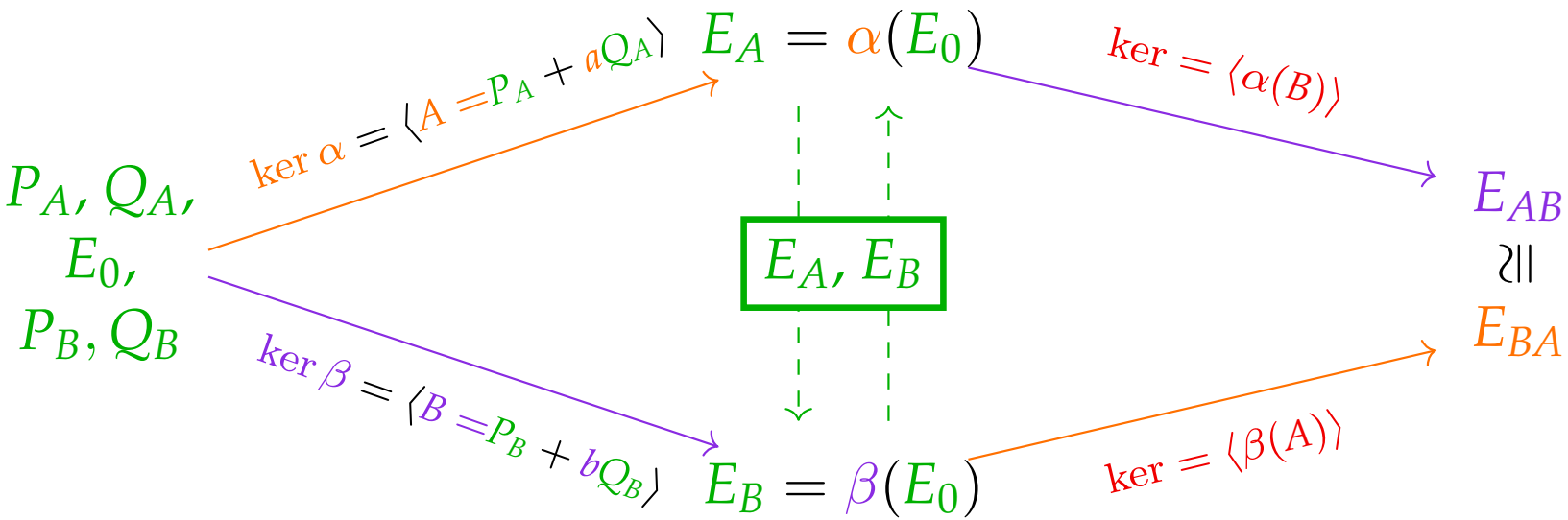

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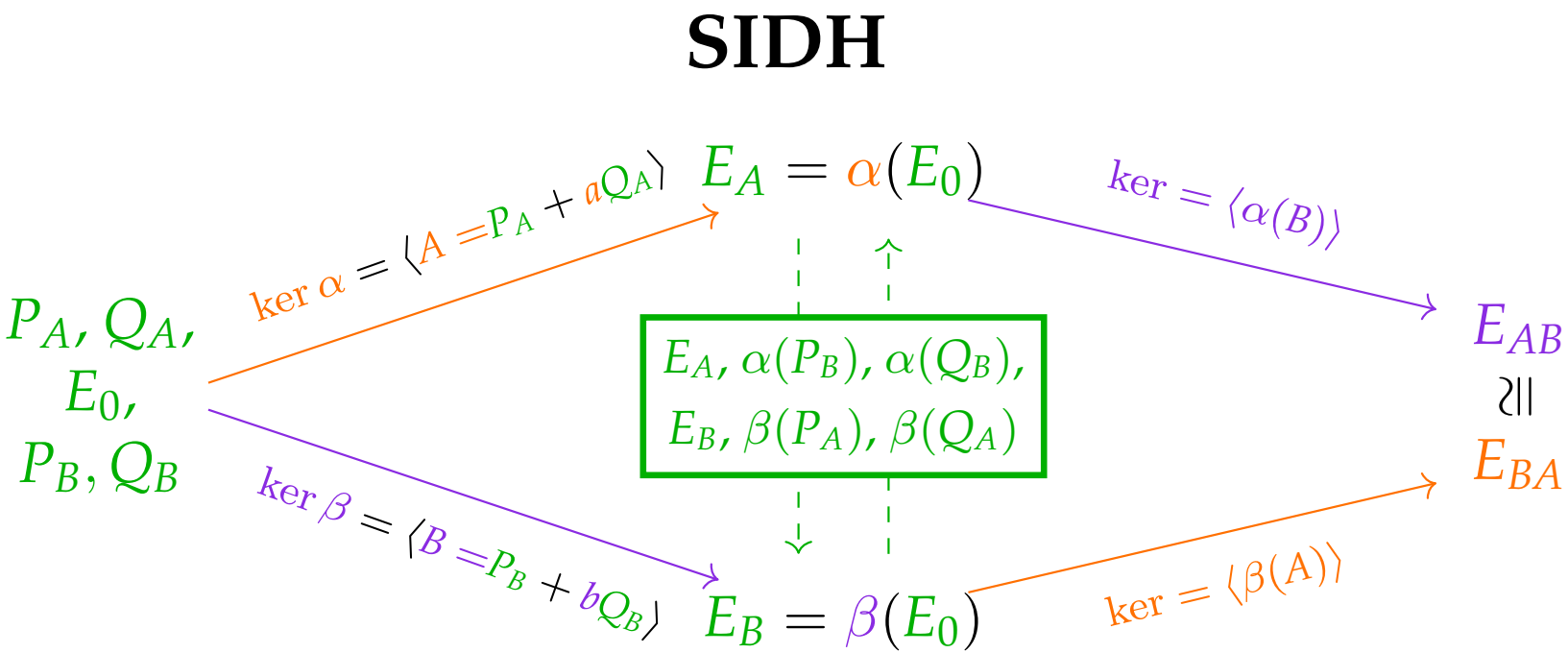

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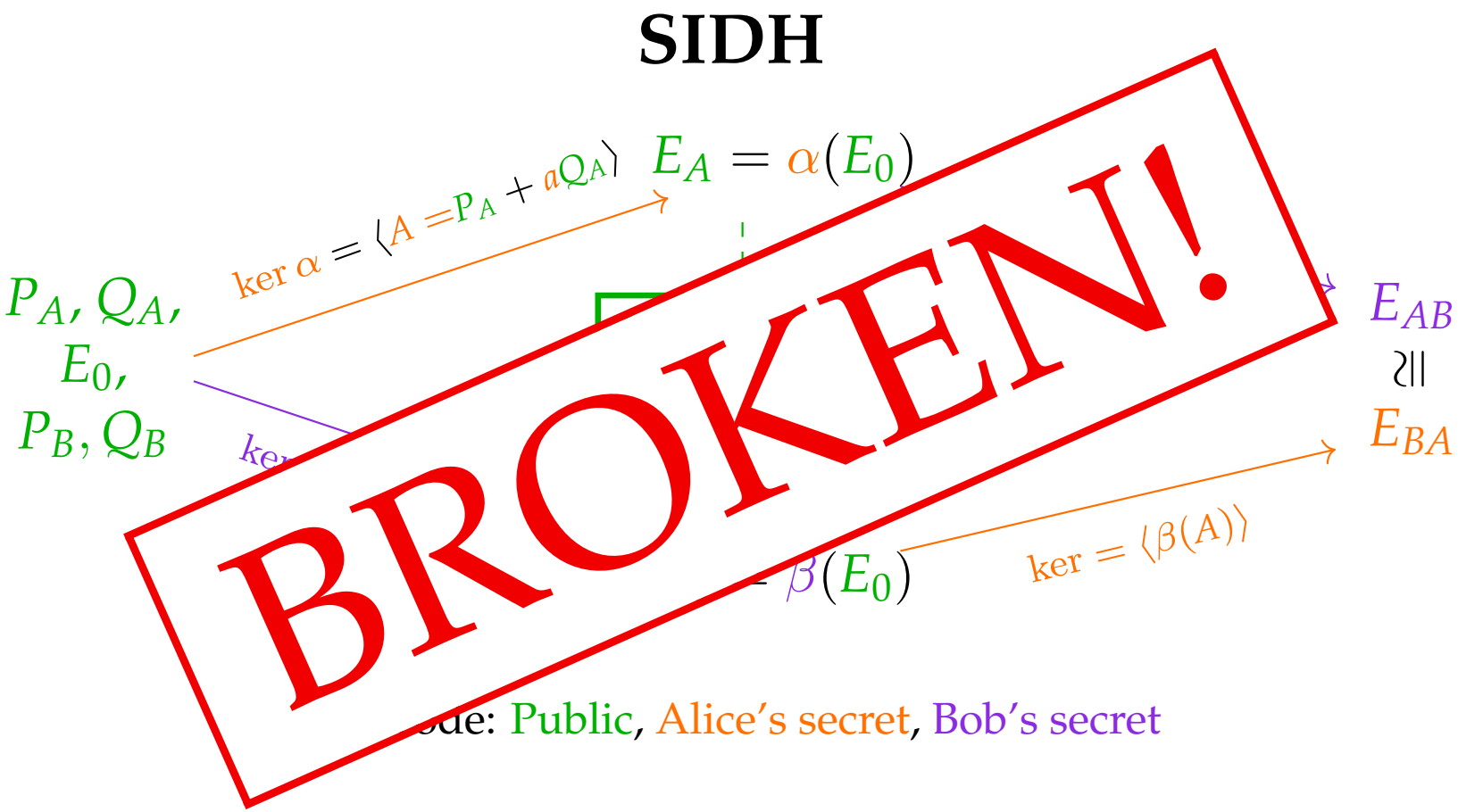

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There are public elliptic curves E_0 and E_A, and a secret isogeny $\alpha : E_0 \to E_A$. Given the points P_B, Q_B on E_0 and $\alpha(P_B)$, $\alpha(Q_B)$, compute α. (modulo technical restrictions)*

*Details for the elliptic curve lovers:

p a large prime; $E_0/\mathbb{F}_{p^2}$ and $E_A/\mathbb{F}_{p^2}$ supersingular; $\deg(\alpha)$, B public large smooth coprime integers; points P_B, Q_B chosen such that $\langle P_B, Q_B \rangle = E_0[B]$.

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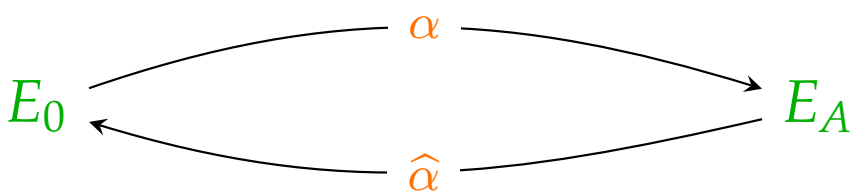

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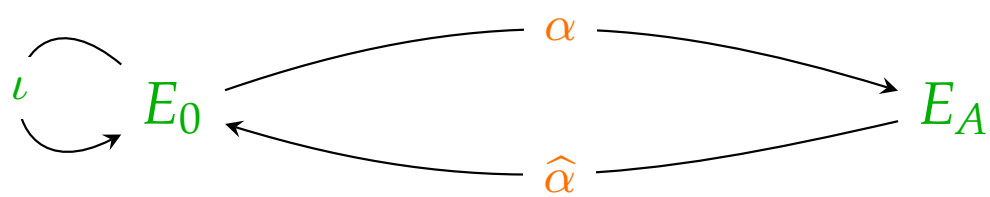

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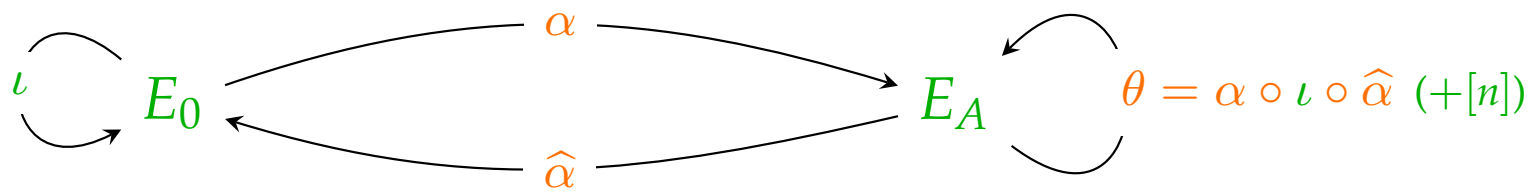

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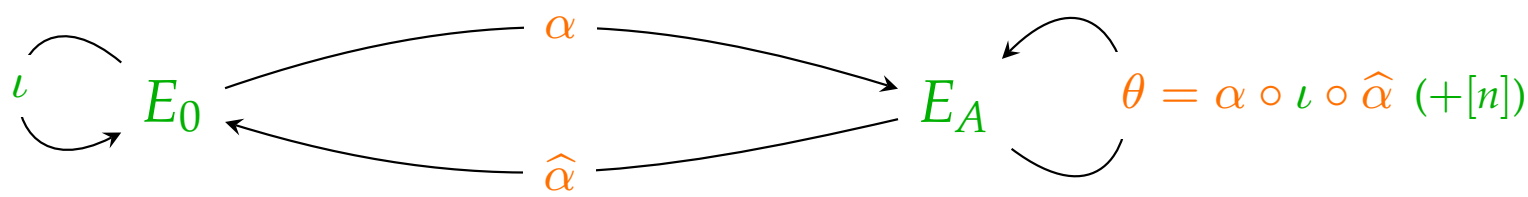

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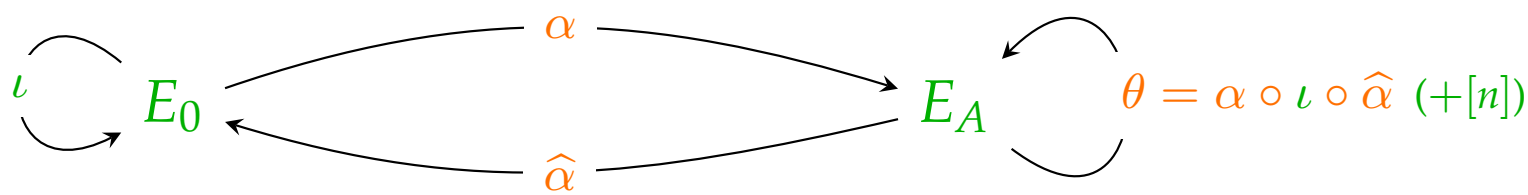

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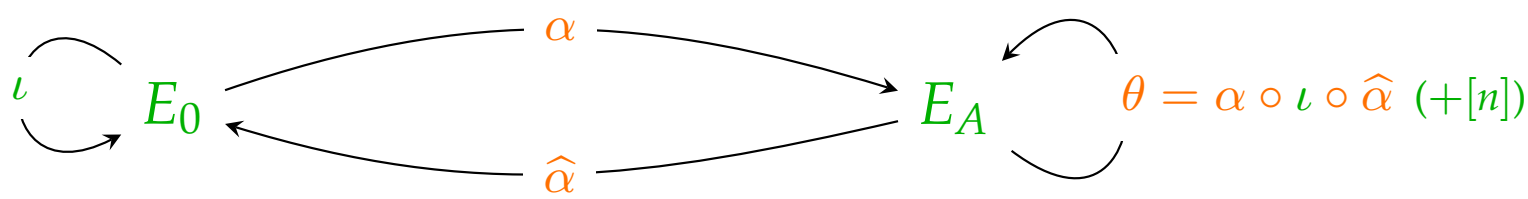

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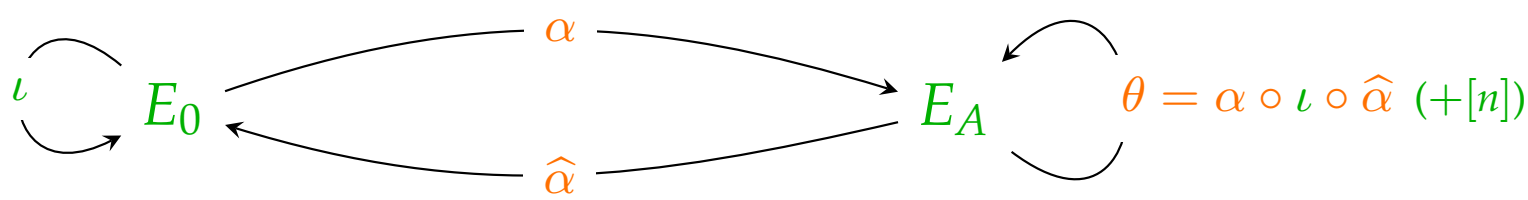

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- Restriction # 2 rules out SIKE parameters, where $B \approx \deg(\alpha)$ (and $p \approx B \cdot \deg \alpha$).

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$\rightsquigarrow$ still not enough. But! Kani's theorem:

- Constructs E_1, E_2 such that there exists a (structure-preserving) isogeny

$$E_1 \times E_A \to E_0 \times E_2$$

  of the right degree, B^2.
- Petit's trick then applies.

Recovering the secret

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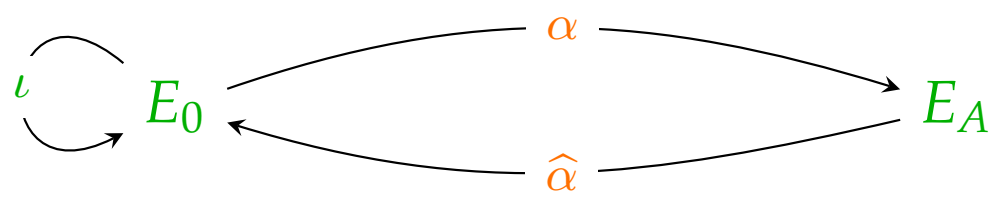

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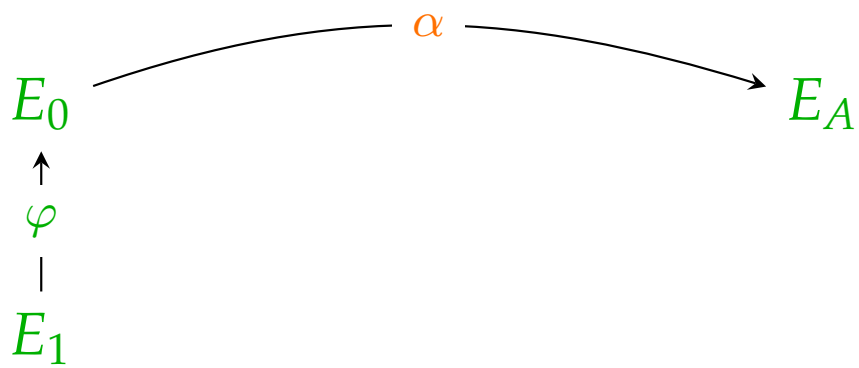

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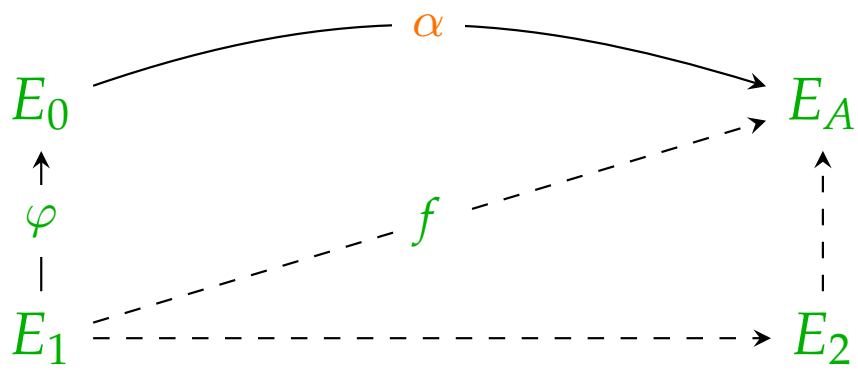

Kani's theorem constructs the above such that

$$\Phi = \begin{pmatrix} \varphi & -\widehat{\alpha} \\ * & * \end{pmatrix} : E_1 \times E_A \to E_0 \times E_2$$

is a structure preserving isogeny of degree B^2, and

$$\ker(\Phi) = \{(\deg(\alpha)P, f(P)) : P \in E_1[B]\}$$

$\rightsquigarrow$ can compute Φ and read off secret α!

Selected applications

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$$E_{\mathrm{pk}}$$

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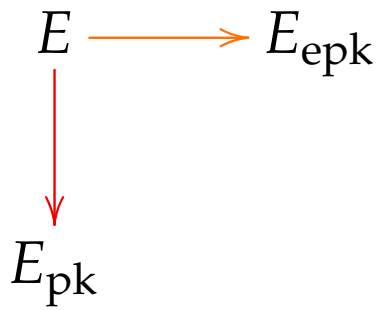

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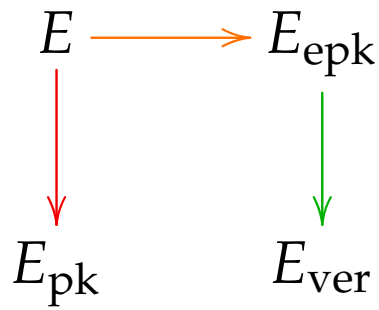

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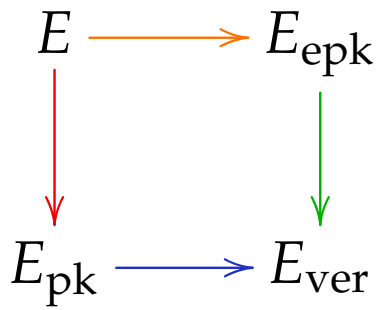

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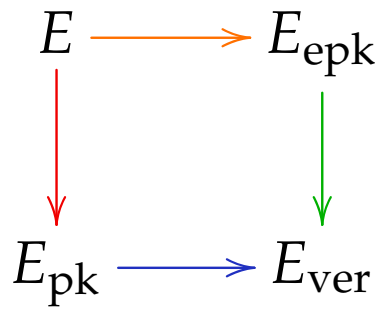

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