Isogeny graphs of abelian varieties and applications to M-SIDH

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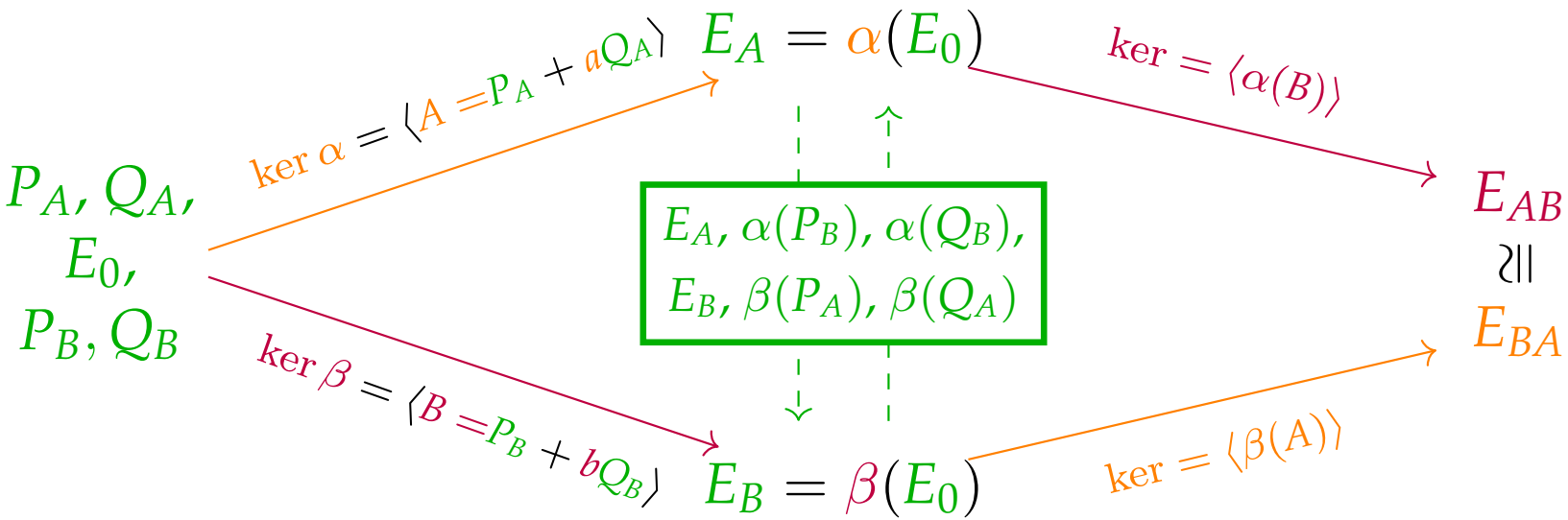
What is this about?

- What is M-SIDH?
- Can we apply SIDH attack methods to M-SIDH?
  - Mostly representing discussions w/ Basso, Cottaar, Fouotsa, Laflamme, Minko, Mocanu, Mokrani, Pedersen, Santos, Stange, Wesolowski
- Theoretical and computational barriers re: isogenies of abelian varieties.
- What do we know?
  - Includes discussions w/ Brooks, Ionica, Jetchev, Milio, Robert, Streng, Vuille, Wesolowski
- Other applications of pushing barriers?

M-SIDH Fouotsa, Moriya, Petit '22

SIDH



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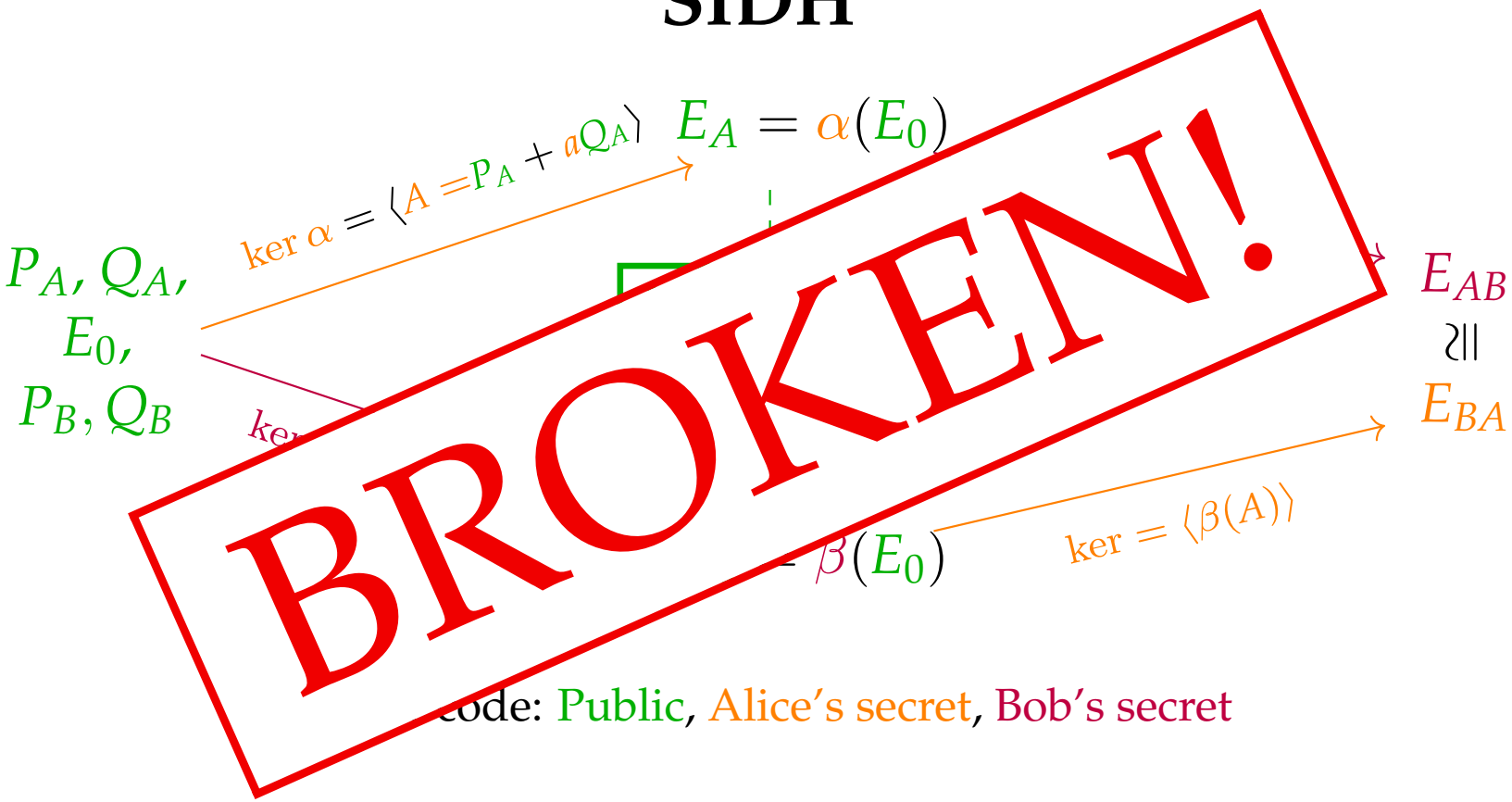

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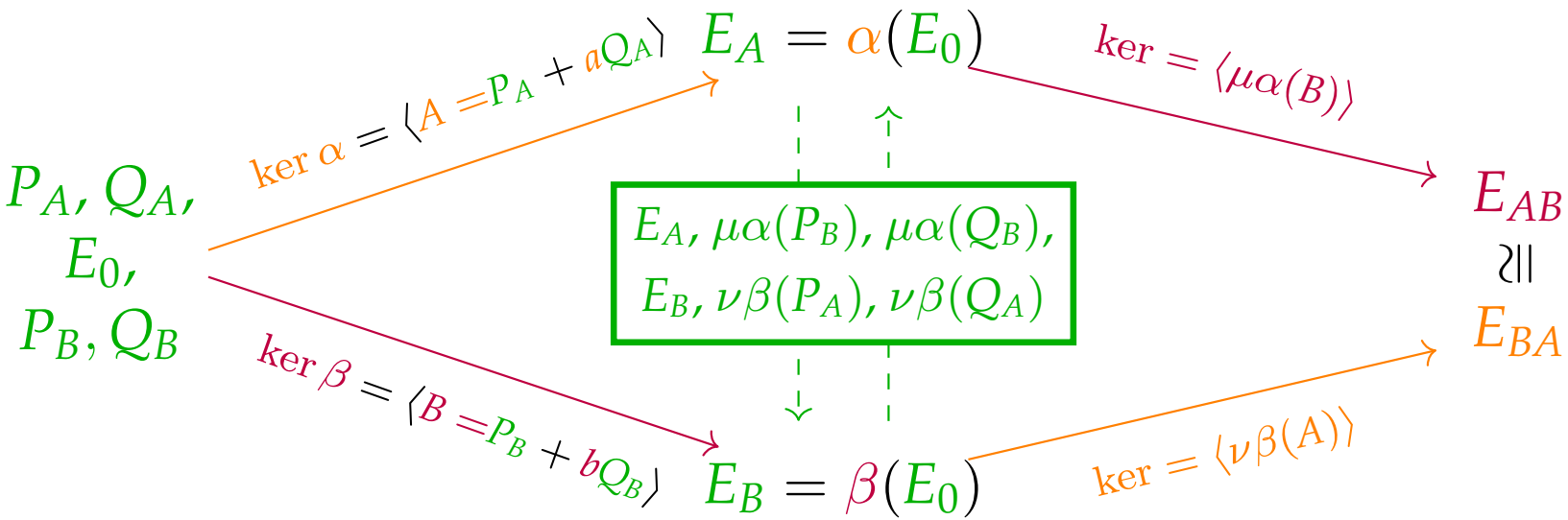

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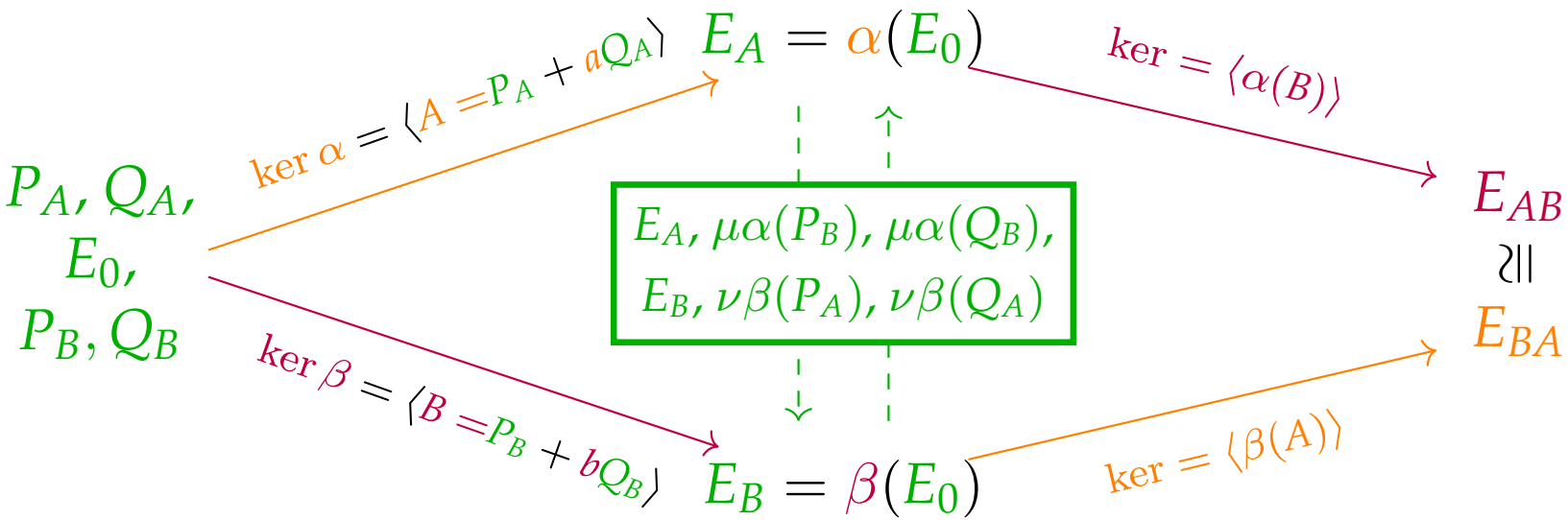

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Petit vs. SIDH: torsion points to isogenies

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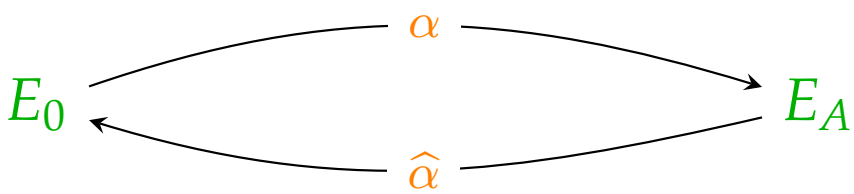

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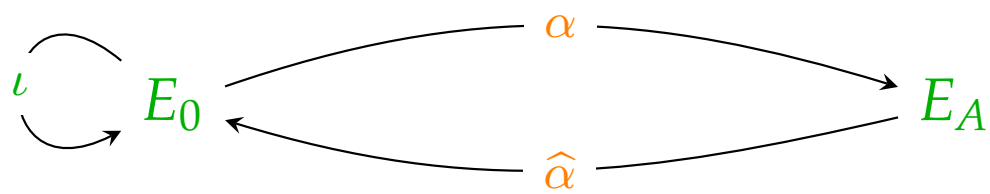

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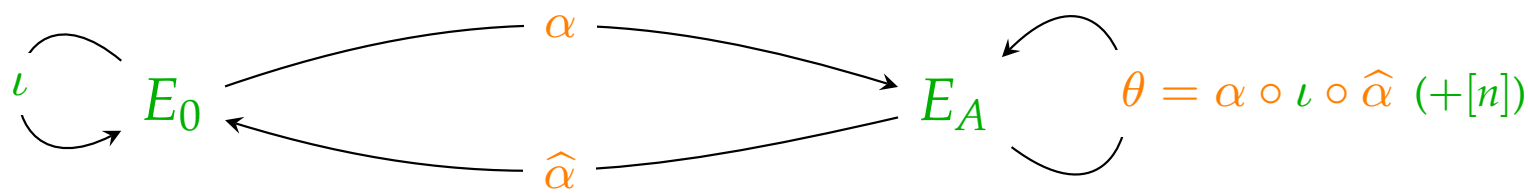

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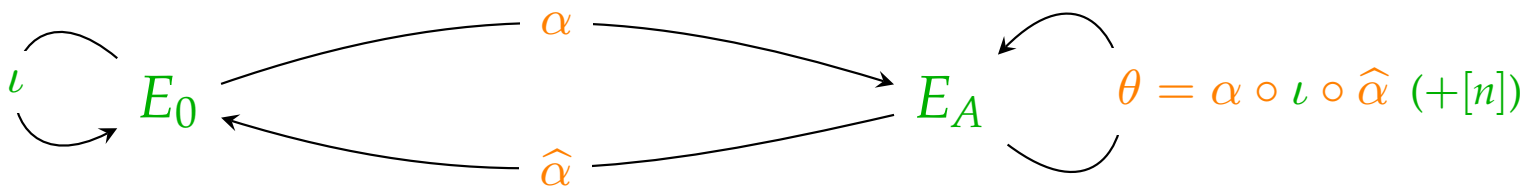

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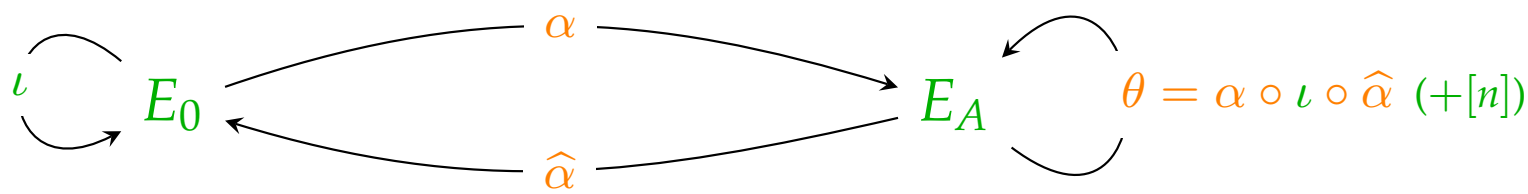

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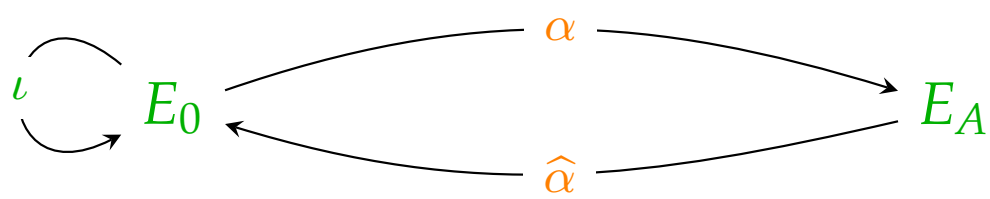

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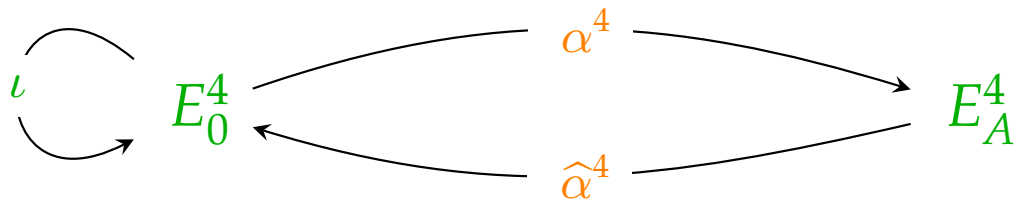

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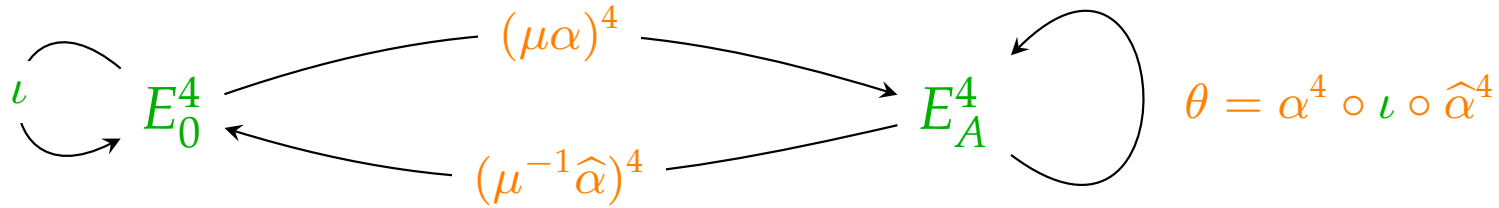

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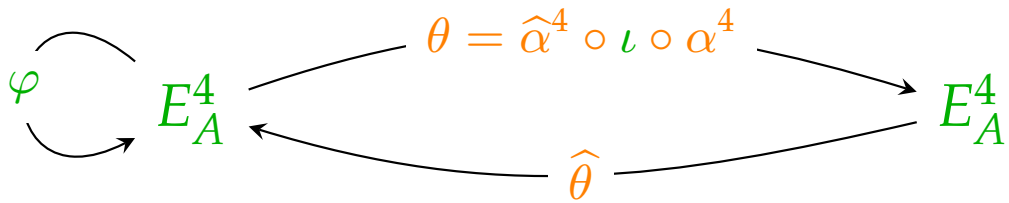

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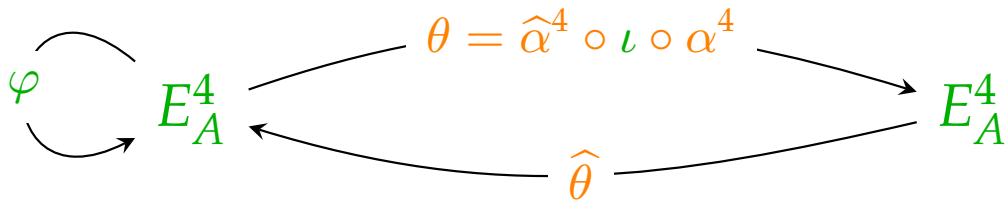

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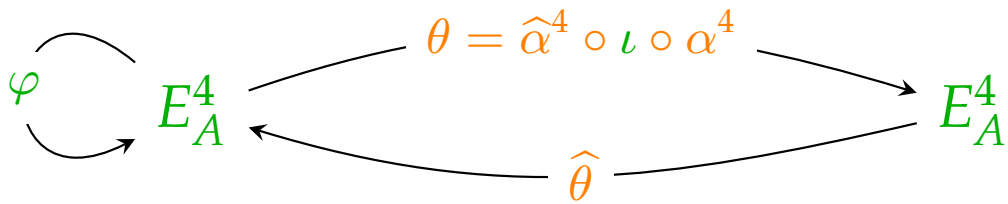

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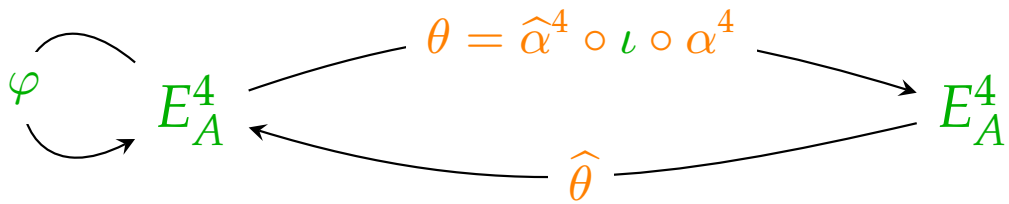

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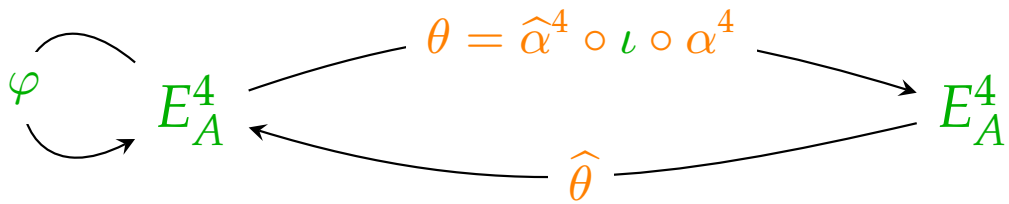

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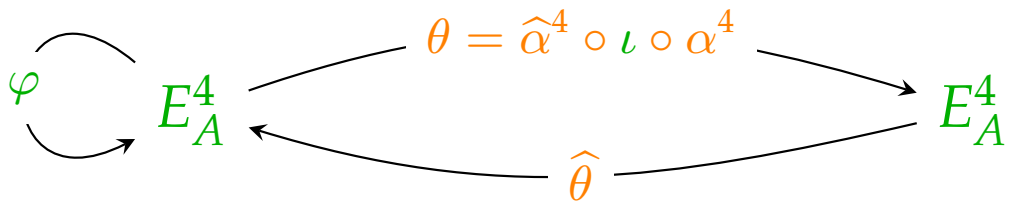

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- Isogeny computation feasible as dimension grows?

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Example: when $g = 2$

$$M = \begin{pmatrix} a & -\iota \\ \widehat{\iota} & b \end{pmatrix},$$

with $a, b \in \mathbb{Z}$ and $\iota \in \mathrm{End}(E) \subseteq B_{p,\infty}$.

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Recall:

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- Now: What are these isogeny subgraphs?

What is known: isogenies in g dimensions

Recall:

Definition

$f : E \to E'$ an isogeny of elliptic curves $/\mathbb{F}_{p^n}$. $p \neq \ell$ prime.

If $\ker(f) \cong \mathbb{Z}/\ell\mathbb{Z}$, we call f an ℓ-isogeny.

If f an ℓ-isogeny, then $f^\vee \circ f = [\ell]$.

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$\deg(f) = \ell^g \rightsquigarrow$

Question: isogenies of degree ℓ when $g > 1$?

What is known: isogenies in g dimensions

Example: The Jacobian $\mathrm{Jac}(C)$ of $C/\mathbb{F}_{17} : y^2 = x^6 + 2x + 1$.
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Do these isogenies always exist?

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- Write $\mathcal{O}_{K_0}$ for the ring of integers of K_0. If:
  1. $\mu \in \mathcal{O}_{K_0}$ is a prime element and is totally positive (all embeddings are positive), $\mathrm{Norm}_{K_0/\mathbb{Q}}(\mu) = \ell$, and
  2. $\mathcal{O}_{K_0} \subseteq \mathrm{End}(A)$,

  then $P \in A[\mu]$, $\mathrm{ord}(P) = \ell$
  $\rightsquigarrow$ cyclic isogeny f with $\ker(f) = \langle P \rangle$

Barriers to breaking M-SIDH

Recall:

- Problem: M-SIDH Attack needs small, nontrivial, known $\iota \in \mathrm{End}(E_0^4)$.
- Idea # 1: orient by small, nontrivial, known $\iota \in \mathrm{End}(E_0^g)$.

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- Problem: M-SIDH Attack needs small, nontrivial, known $\iota \in \mathrm{End}(E_0^4)$.
- Idea # 1: orient by small, nontrivial, known $\iota \in \mathrm{End}(E_0^g)$.
- What are the oriented isogeny graphs of abelian varieties A with $\iota \in \mathrm{End}(A)$?

What is known*: oriented isogeny subgraphs

Recall:
An ℓ-isogeny graph of elliptic curves $/k$ has:

- Vertices: Elliptic curves E/k (up to isomorphism).
- Edges: An edge $E - E'$ represents an ℓ-isogeny $E \to E'$ and its dual (up to isomorphism).

What is known*: oriented isogeny subgraphs

An $(\ell, \ldots, \ell)$-isogeny (resp. cyclic μ-isogeny) graph of abelian varieties $/k$ satisfying property P [1] has:

- Vertices: Abelian varieties A/k satisying P (up to P-preserving-isomorphism).
- Edges: An edge $A - A'$ represents an P-preserving $(\ell, \ldots, \ell)$-isogeny (resp. cyclic μ-isogeny) $A \to A'$ and its dual (up to P-preserving-isomorphism).

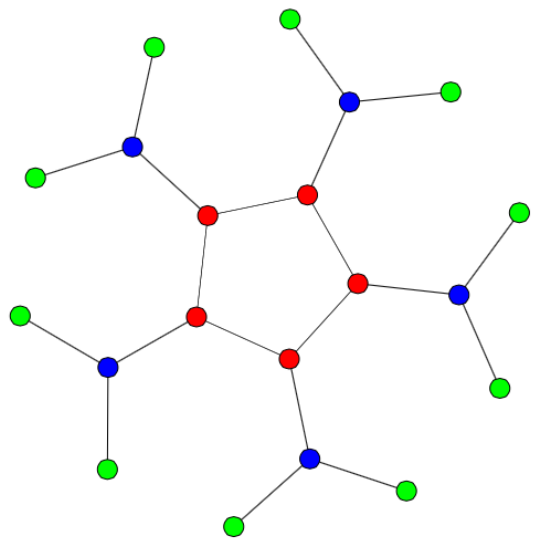
[1] This property should include that abelian varieties are isomorphic to their duals

What is known*: oriented isogeny subgraphs

Recall:

Theorem ([K96])

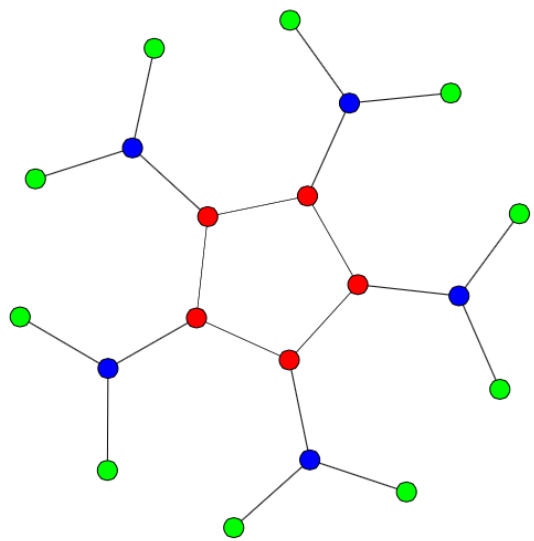
Let $E/\mathbb{F}_q$ be an ordinary elliptic curve such that $j(E) \neq 0, 1728$, and let $p \neq \ell \in \mathbb{Z}$ be a prime. Then the connected component of the ℓ-isogeny graph containing E is a volcano.



What is known*: oriented isogeny subgraphs

Theorem ([BJW17]* / [M18]*)

Let $A/\mathbb{F}_q$ be a principally polarised abelian variety, ι, μ as before (modulo technical conditions) If $\mathcal{O}_{K_0} \subseteq \mathrm{End}(A)$, then the connected component of the ι-oriented cyclic μ-isogeny graph containing A is a volcano.



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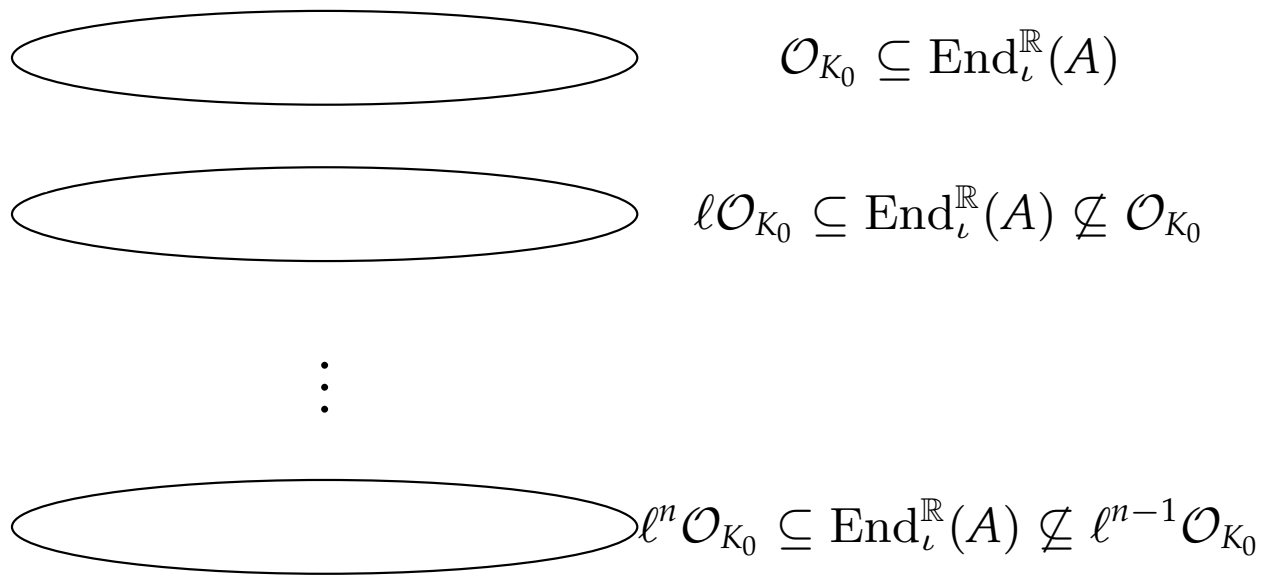
Connected component of ι-oriented $(\ell, \ldots, \ell)$-isogeny graph of a principally polarised abelian variety over $\mathbb{F}_q$:

$\mathrm{End}_\iota^{\mathbb{R}}(A)$ is the real part of $\mathrm{End}(A) \cap \mathbb{Q}(\iota)$.



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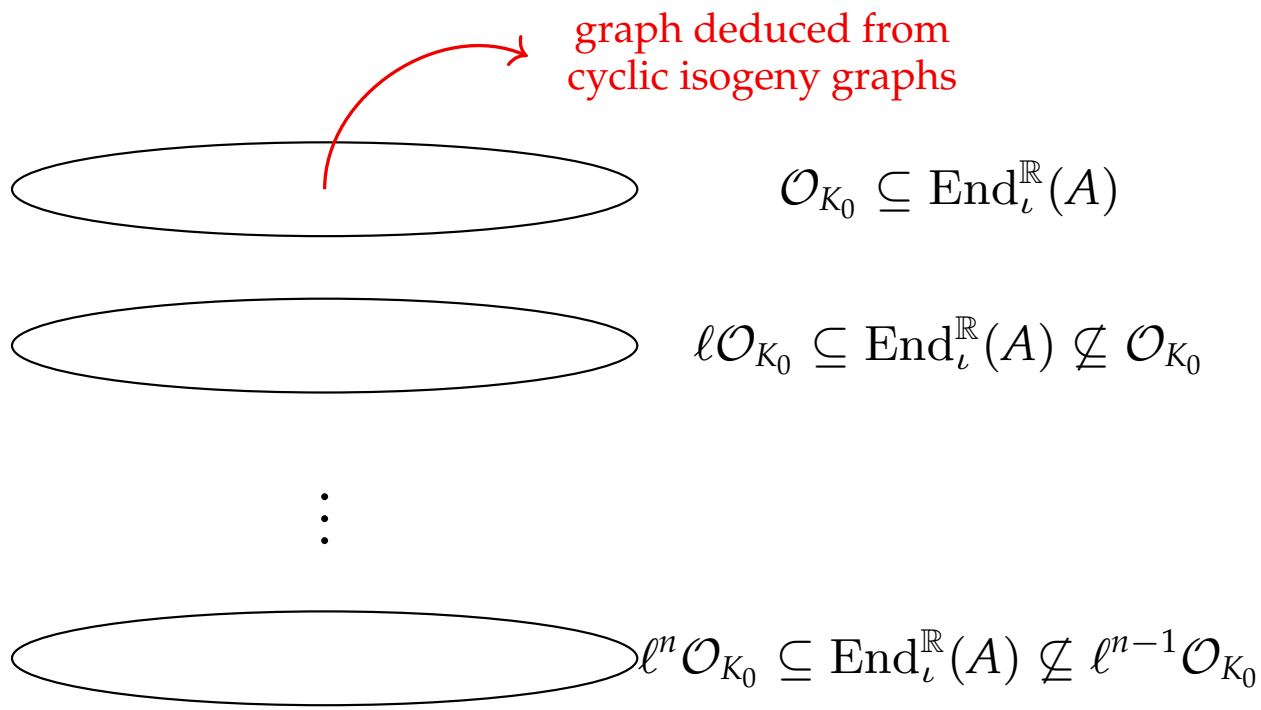

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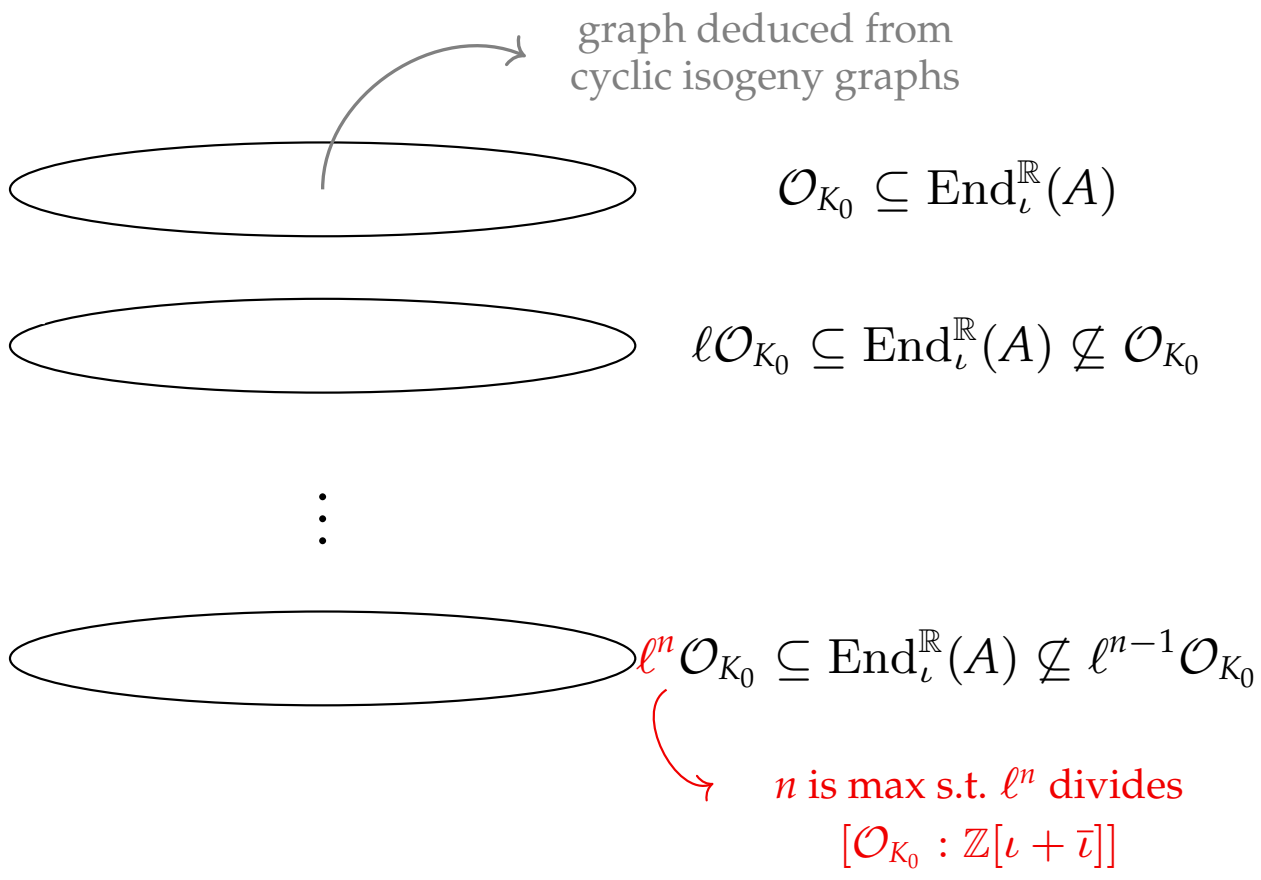

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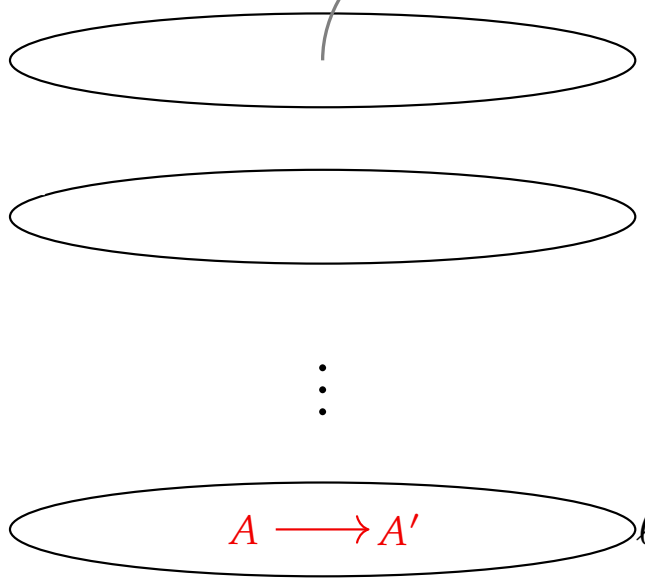
graph deduced from cyclic isogeny graphs



Isogenies within layers are 'RM-horizontal'

n is max s.t. ℓ^n divides $[\mathcal{O}_{K_0} : \mathbb{Z}[\iota + \bar{\iota}]]$

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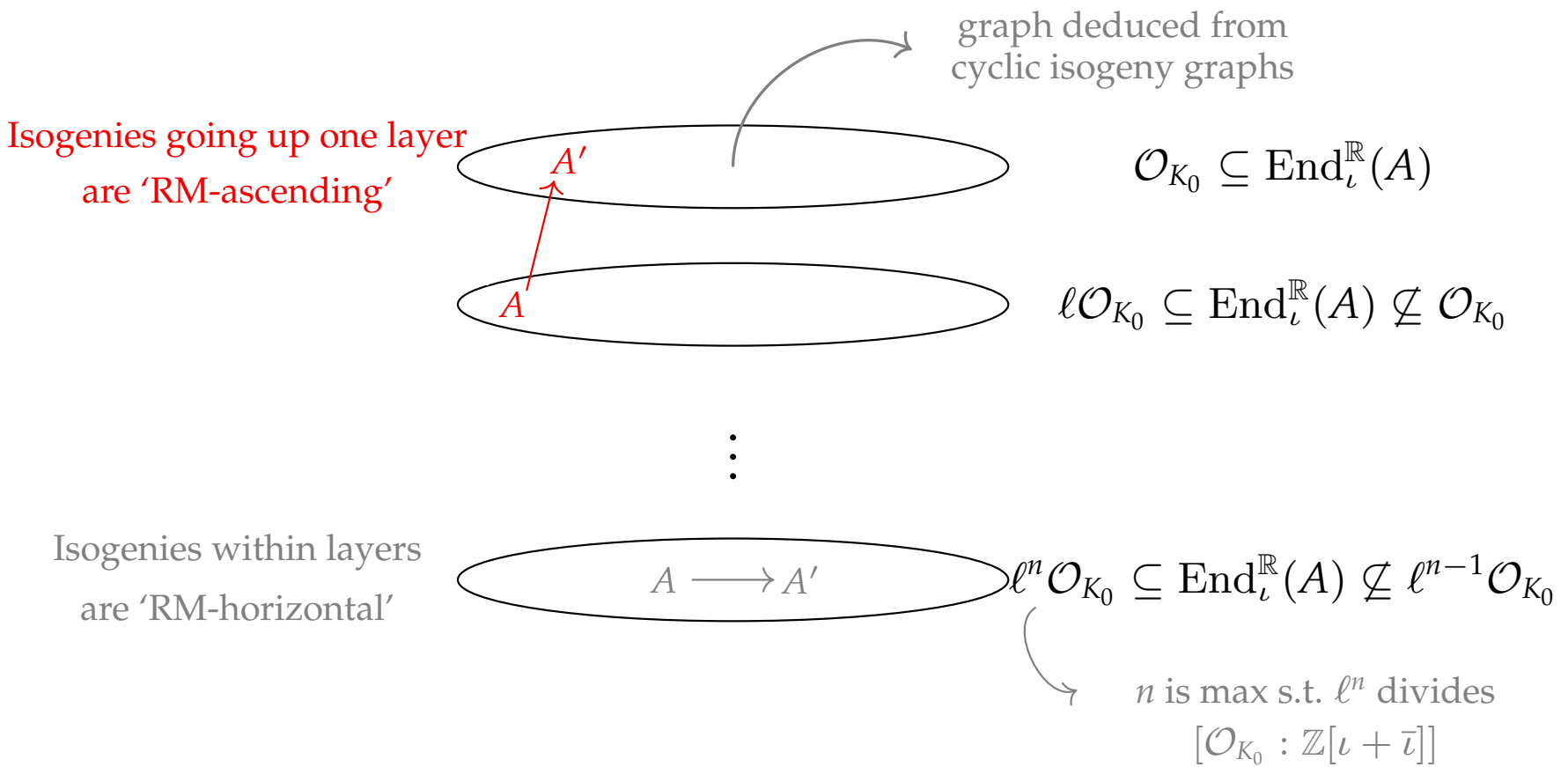

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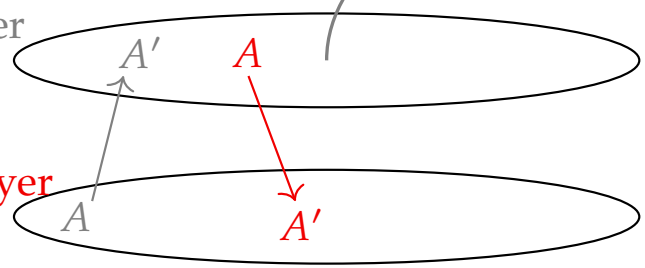

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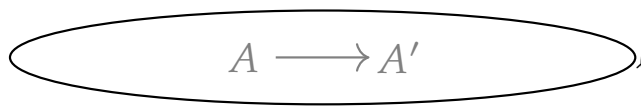

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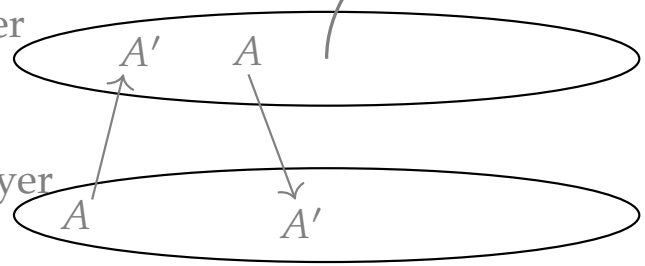

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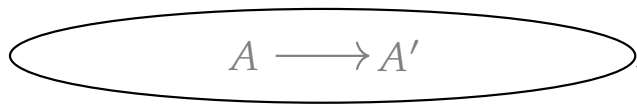

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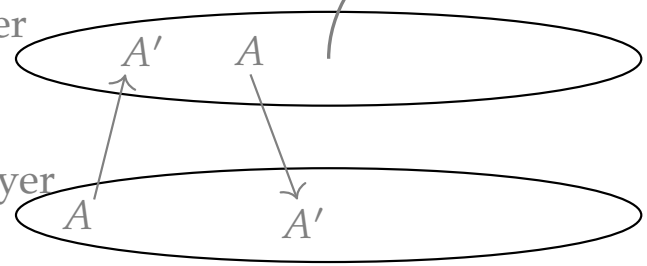

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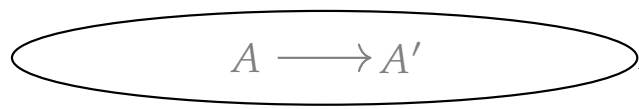

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- Idea # 2: use nonprincipal endomorphisms.
  - No existing literature on computation.

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  Potential application: easier isogeny problem?

Thank you!