Elliptic-curve and isogeny-based cryptography

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Summer School on real-world crypto and privacy
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Why elliptic-curve cryptography (ECC)?

ECC is widely deployed across many use cases. Why? It is:

- Low memory
- Fast
- Flexible
  - TLS, AKE, Signal protocol, IBE (using pairings), . . .
- Robust

Ex: WhatsApp (uses Signal protocol)

Public Key Types

- `Identity Key Pair` – A long-term Curve25519 key pair, generated at install time.
- `Signed Pre Key` – A medium-term Curve25519 key pair, generated at install time, signed by the `Identity Key`, and rotated on a periodic timed basis.
- `One-Time Pre Keys` – A queue of Curve25519 key pairs for one time use, generated at install time, and replenished as needed.

Session Key Types

- `Root Key` – A 32-byte value that is used to create `Chain Keys`.
- `Chain Key` – A 32-byte value that is used to create `Message Keys`.
- `Message Key` – An 80-byte value that is used to encrypt message contents. 32 bytes are used for an AES-256 key, 32 bytes for a HMAC-SHA256 key, and 16 bytes for an IV.

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  - eg. $(3 \pmod 5)^2 = 3 \cdot 3 \pmod 5$.

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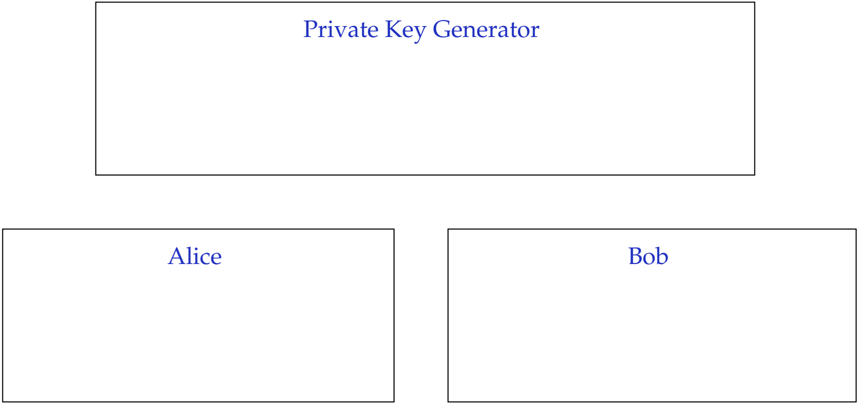
Why is this useful?

Pairings in (simplified) IBE (Boneh-Franklin)

Scenario: Bob authenticates an anonymous Alice.



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Private Key Generator

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  - Instances of the Weil pairing can be efficiently computed with Miller's algorithm.

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- for most pairing instances, p is special, giving $L_{p^k}(1/3, c)$

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$g \in E(\mathbb{F}_{p^k})$: $\underbrace{g * \cdots * g}_{n \text{ times}} \longrightarrow$ **Pollard-ρ** $\longrightarrow n$

Complexity: if p large, non-special, and k small, for most E,
$O(\sqrt{r})$, where r is the largest prime dividing $\#E(\mathbb{F}_{p^k})$

Pairing-friendly families

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- Disclaimer for papers before 2016: New improvements/refinements to the attack methods in 2016. See eg. [BD17] for an overview.
  - Worst-case asymptotic complexity went from $L_{p^k}[1/3, 1.923]$ to $L_{p^k}[1/3, 1.526]$.

That’s cute, but what about quantum computers?

Cryptography

Sender | Channel with eavesdropper 'Eve' | Receiver

Cryptography

Sender — Channel with eavesdropper ‘Eve’ — Receiver

Problems:

- Communication channels (adversaries) store and spy on our data
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Goals:

- Confidentiality despite Eve’s espionage.
- Integrity: recognising Eve’s espionage.

(Slide mostly stolen from Tanja Lange)

Post-quantum cryptography

Sender — Channel with eavesdropper 'Eve' — Receiver

Post-quantum cryptography

Sender　　Channel with eavesdropper 'Eve'　　Receiver

- Eve has a quantum computer.
- Harry and Meghan don't have a quantum computer.

(Slide mostly stolen from Tanja Lange)

Why does Eve need a quantum computer?

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> Main goal: replace the use of the discrete logarithm problem in asymmetric cryptography with something quantum-resistant.

Where are we now?

- Post-quantum cryptography discussion dominated by NIST competition for standardization.

Where are we now?

- Post-quantum cryptography discussion dominated by NIST competition for standardization.
- This initiative comes after a US report with:

Key Finding 10: Even if a quantum computer that can decrypt current cryptographic ciphers is more than a decade off, the hazard of such a machine is high enough—and the time frame for transitioning to a new security protocol is sufficiently long and uncertain—that prioritization of the development, standardization, and deployment of post-quantum cryptography is critical for minimizing the chance of a potential security and privacy disaster.

Recall: Diffie–Hellman key exchange ’76

Public parameters:

- a prime p (experts: uses $\mathbb{F}_p^*$, today also elliptic curves)
- a number $g \pmod{p}$ (nonexperts: think of an integer less than p)

Alice	Eve	Bob
$\mathrm{sk}_A \xleftarrow{\text{random}} \{0 \ldots p-1\}$		$\mathrm{sk}_B \xleftarrow{\text{random}} \{0 \ldots p-1\}$
$\mathrm{pk}_A = g^{\mathrm{sk}_A}$ →		← $\mathrm{pk}_B = g^{\mathrm{sk}_B}$
$ss := (g^{\mathrm{sk}_B})^{\mathrm{sk}_A}$		$ss := (g^{\mathrm{sk}_A})^{\mathrm{sk}_B}$

- Alice and Bob agree on a shared secret key ss, then they can use that to encrypt their messages.
- Eve sees $\mathrm{pk}_A = g^{\mathrm{sk}_A}$, $\mathrm{pk}_B = g^{\mathrm{sk}_B}$; can’t find sk_A, sk_B, ss.

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Alice — Eve

$\mathrm{sk}_A \xleftarrow{\text{random}} \{0 \ldots p-1\}$ … $-1\}$

$\mathrm{pk}_A = g^{\mathrm{sk}_A}$ … $\mathrm{pk}_B = g^{\mathrm{sk}_B}$

$ss := (g^{\mathrm{sk}_A})^{\mathrm{sk}_B}$

Broken by Shor!

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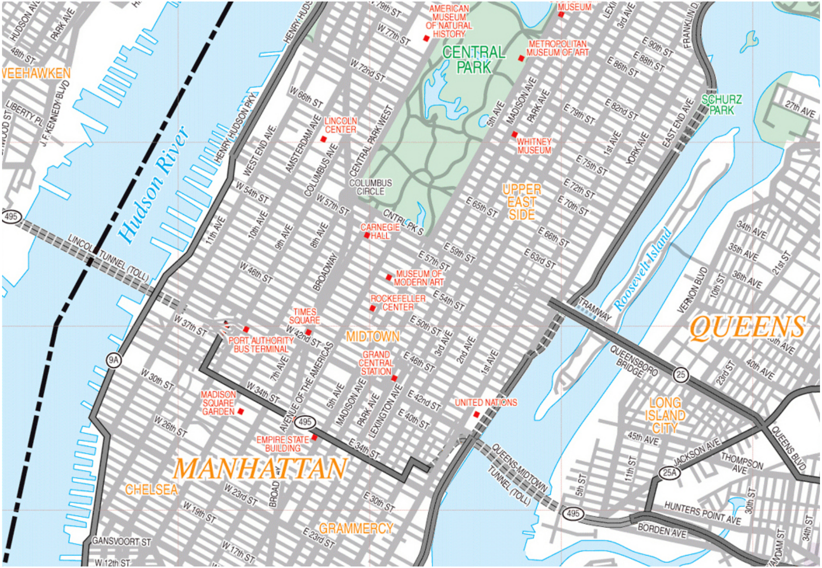
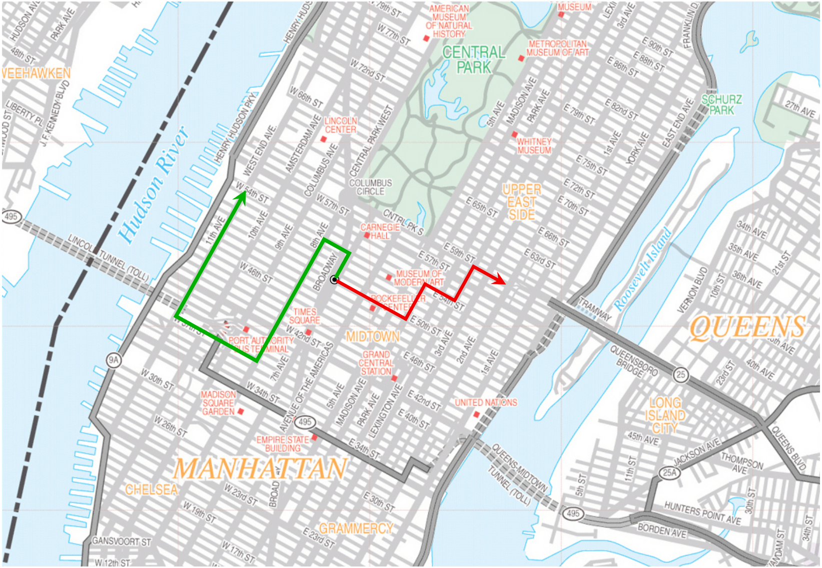
- Multivariate signatures: based on solving simulateneous multivariate equations.
  Short signatures, large public keys, slow.

Graph walking Diffie–Hellman?



Graph walking Diffie–Hellman?



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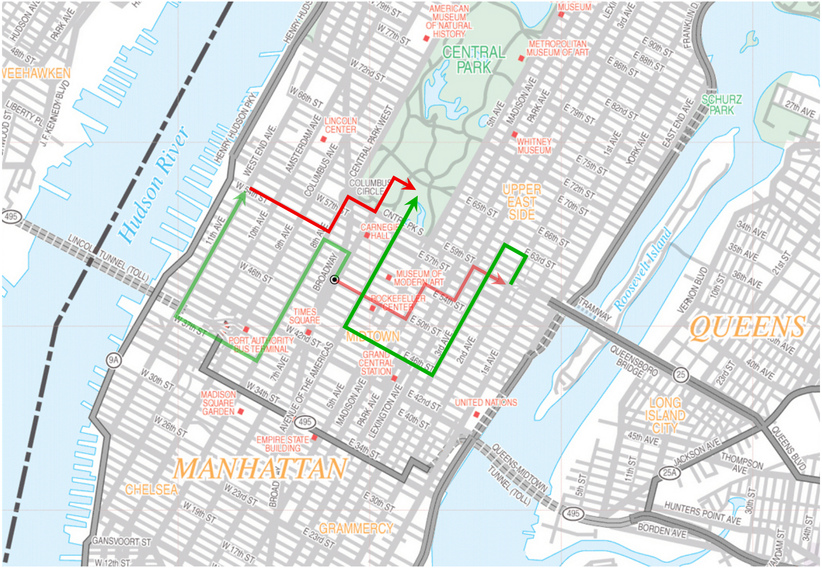

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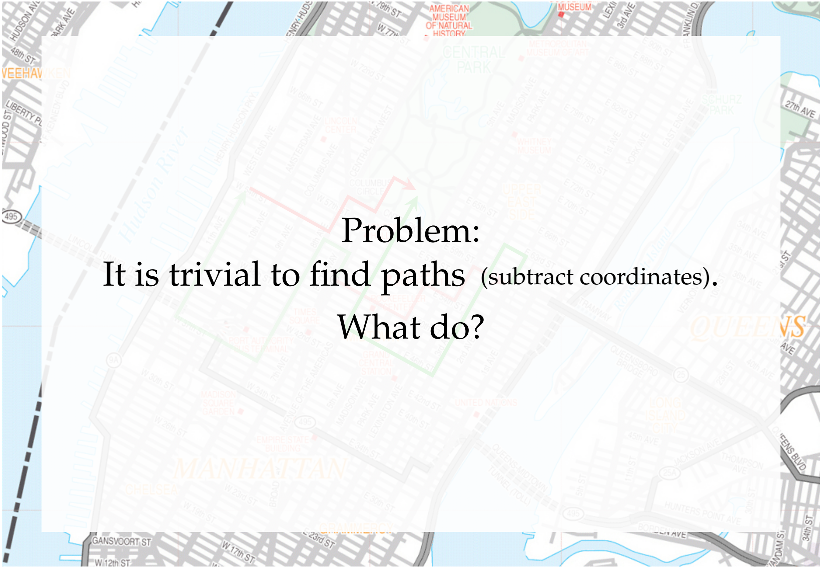

Problem:

It is trivial to find paths (subtract coordinates).

What do?

Big picture

- Isogenies are a source of exponentially-sized graphs.

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It is easy to construct graphs that satisfy *almost* all of these — not enough for crypto!

Stand back!

We’re going to do maths.

Maths background #1/3: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- given by rational functions.
- a group homomorphism.

The degree of a separable* isogeny is the size of its kernel.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m]\colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota\colon (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$

defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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The ring of endomorphisms of E is denoted by $\mathrm{End}(E)$.

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Each isogeny $\varphi\colon E \to E'$ has a unique dual isogeny $\widehat{\varphi}\colon E' \to E$ characterized by $\widehat{\varphi} \circ \varphi = \varphi \circ \widehat{\varphi} = [\deg \varphi]$.

Maths background #2/3: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G : E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

[1](up to isomorphism of E')

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Vélu operates in the field where the points in G live.

$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$

$\rightsquigarrow$ this is why we use supersingular curves!

[1](up to isomorphism of E')

Math slide #3/3: Supersingular isogeny graphs

Let p be a prime, q a power of p, and ℓ a positive integer $\notin p\mathbb{Z}$.

An elliptic curve $E/\mathbb{F}_q$ is *supersingular* if $p \mid (q + 1 - \#E(\mathbb{F}_q))$.

We care about the cases $\#E(\mathbb{F}_p) = p + 1$ and $\#E(\mathbb{F}_{p^2}) = (p + 1)^2$.

$\rightsquigarrow$ easy way to control the group structure by choosing p!

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Let $S \not\ni p$ denote a set of prime numbers.

The supersingular S-isogeny graph over $\mathbb{F}_q$ consists of:

- vertices given by isomorphism classes of supersingular elliptic curves,
- edges given by equivalence classes[1] of ℓ-isogenies ($\ell \in S$),

both defined over $\mathbb{F}_q$.

[1] Two isogenies $\varphi\colon E \to E'$ and $\psi\colon E \to E''$ are identified if $\psi = \iota \circ \varphi$ for some isomorphism $\iota\colon E' \to E''$.

The beauty and the beast

Components of the isogeny graphs look like this:

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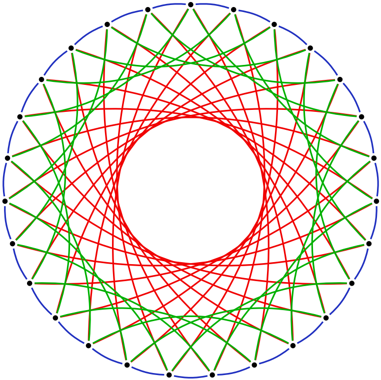

$S = \{3, 5, 7\}$, $q = 419$

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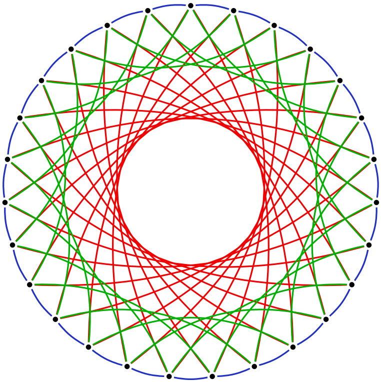

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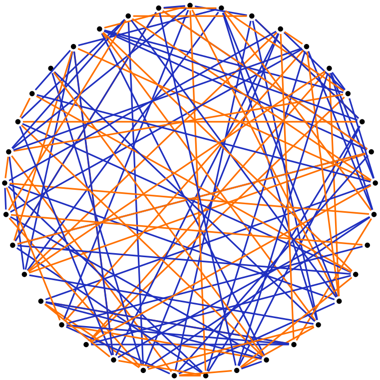
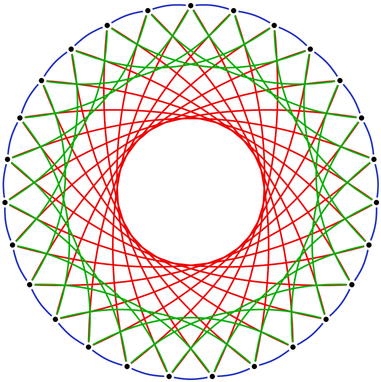


$S = \{2, 3\}$, $q = 431^2$

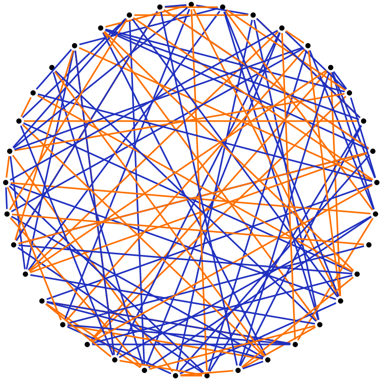
The beauty and the beast

For key exchange/KEM, there are <u>two families</u> of systems:



$q = p$

CSIDH [ˈsiːˌsaɪd]
`https://csidh.isogeny.org`

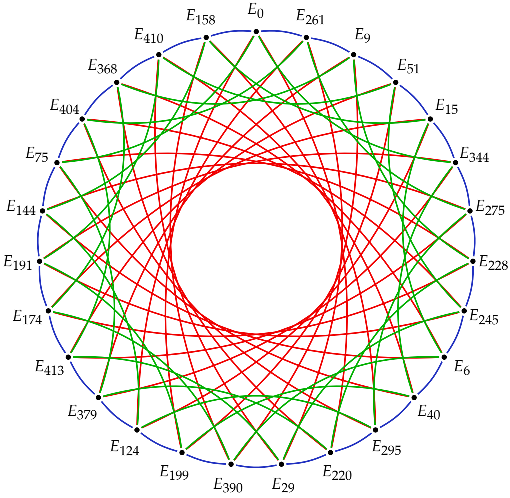


$q = p^2$

SIDH
`https://sike.org`


[ˈsiːˌsaɪd]

Isogeny graphs at the CSIDH



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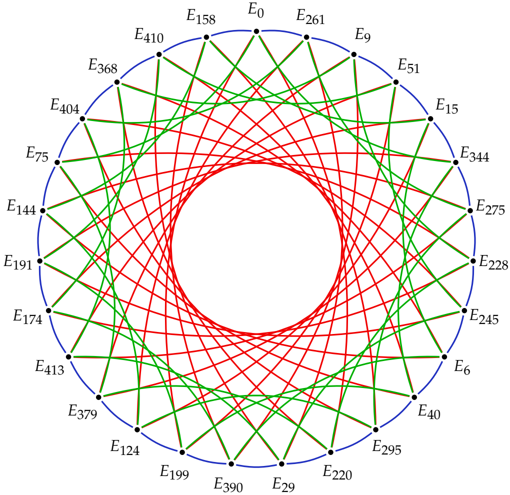

Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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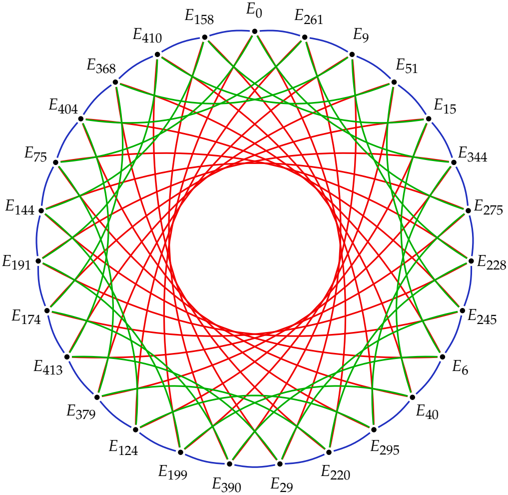


Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies.

Quantumifying Exponentiation

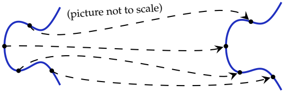
- Idea to replace DLP: replace exponentiation

$$\begin{array}{ccc} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x \end{array}$$

  by a group action on a set.
- Replace G by the set S of supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
- Replace $\mathbb{Z}$ by a commutative group H that acts via isogenies.
- The action of $h \in H$ on S moves the elliptic curves one step around one of the cycles.

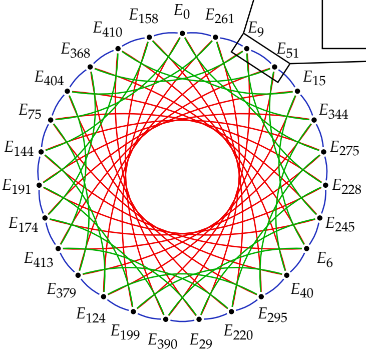
Graphs of elliptic curves

A 3-isogeny



$$E_{51}: y^2 = x^3 + 51x^2 + x \longrightarrow E_9: y^2 = x^3 + 9x^2 + x$$

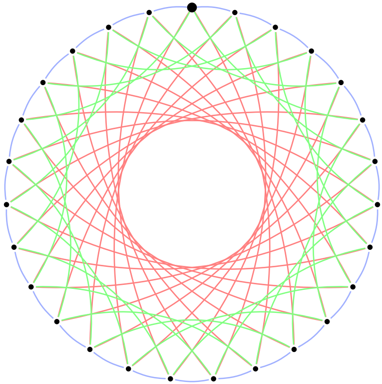
$$(x, y) \longmapsto \left(\frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97}, \; y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$$



Diffie and Hellman go to the CSIDH

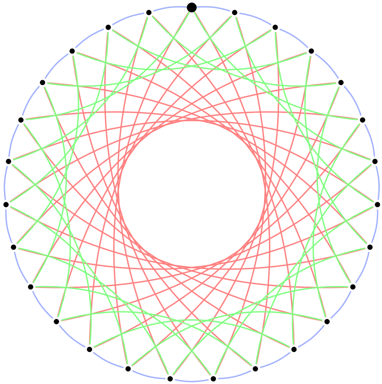
Alice

[+, −, +, −]



Bob

[+, +, −, +]



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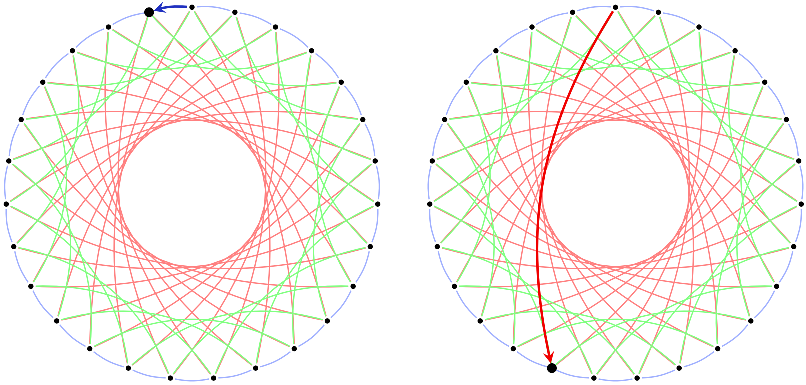

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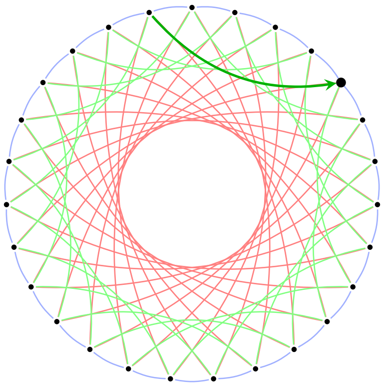

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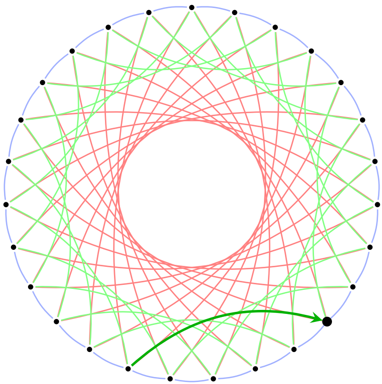

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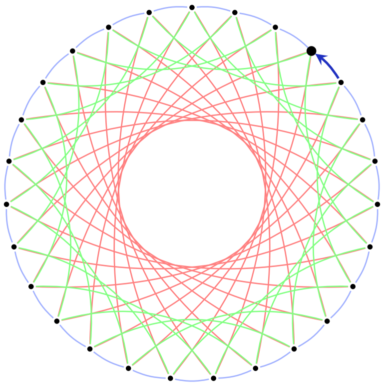
$\uparrow$



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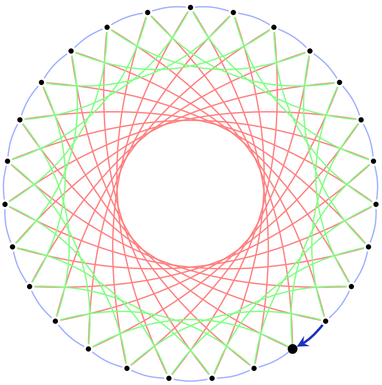

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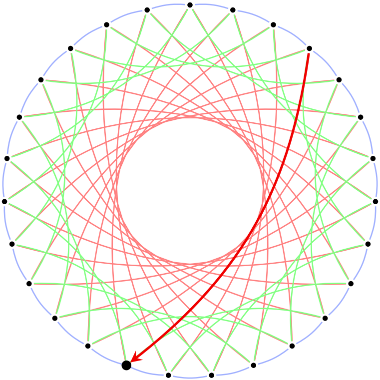

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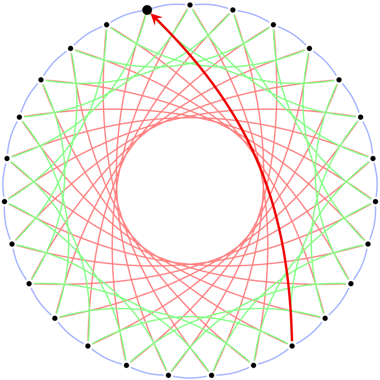

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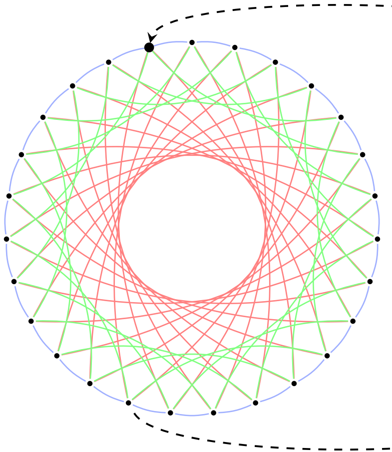

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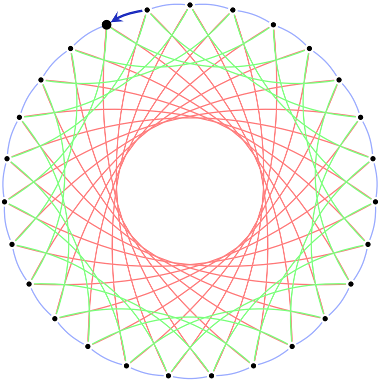

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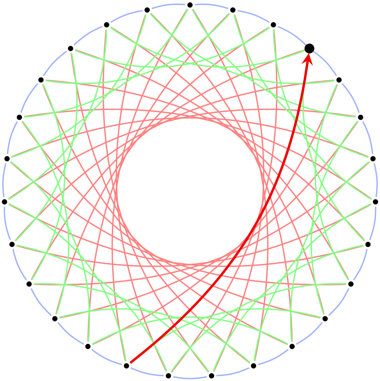

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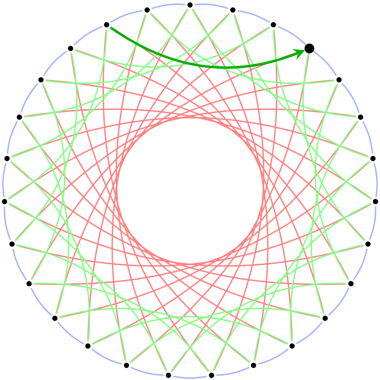

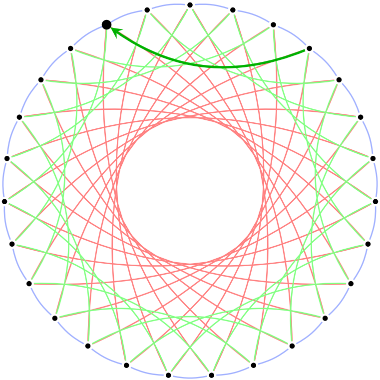

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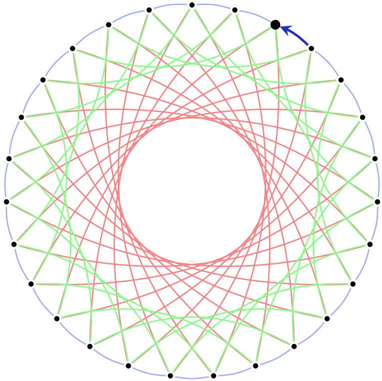

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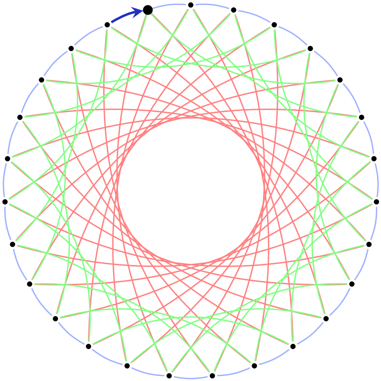

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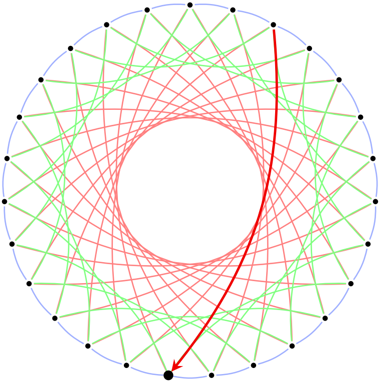

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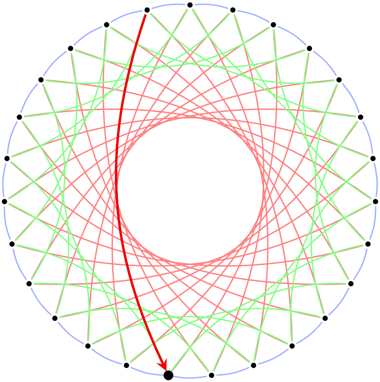

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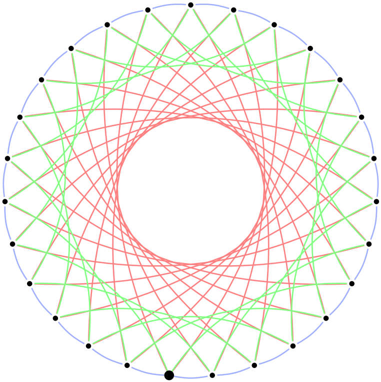

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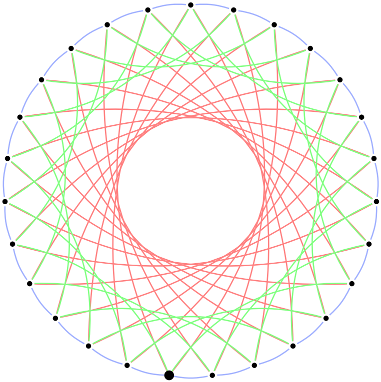

Compute neighbours in the graph

To compute a neighbour of E, we have to compute an ℓ-isogeny from E. To do this:

- Find a point P of order ℓ on E.

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p + 1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p + 1]P = \infty$.[1]

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Original proposal in 2018 paper: $\mathbb{F}_p \approx 512$ bits.

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
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- Overheads from error correction, high quantum memory etc., not yet understood.

Venturing beyond the CSIDH

A selection of advances since original publication (2018):

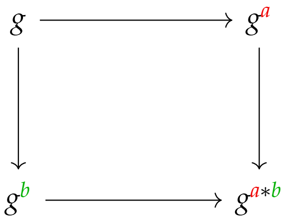
- CSURF [CD19]: exploiting 2-isogenies.
- sqrtVelu [BDLS20]: square-root speed-up on computation of large-degree isogenies.
- Radical isogenies [CDV20]: significant speed-up on isogenies of small-ish degree.
- Some work on different curve forms (e.g. Edwards, Huff).
- Knowledge of $\mathrm{End}(E_0)$ and $\mathrm{End}(E_A)$ breaks CSIDH in classical polynomial time [Wes21].
- The SQALE of CSIDH [CCJR22]: carefully constructed CSIDH parameters less susceptible to Kuperberg's algorithm.
- CTIDH [$B^2C^2LMS^2$]: Efficient constant-time CSIDH-style construction.

Now:

SIDH

Supersingular Isogeny Diffie–Hellman

Diffie-Hellman: High-level view



SIDH: High-level view

$$\begin{array}{ccc} E & \xrightarrow{\varphi_A} & E/A \\ {\scriptstyle \varphi_B}\downarrow & & \downarrow{\scriptstyle \varphi_{B'}} \\ E/B & \xrightarrow[\varphi_{A'}]{} & E/\langle A, B\rangle \end{array}$$

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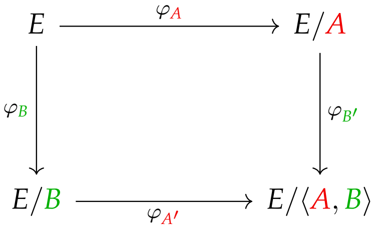

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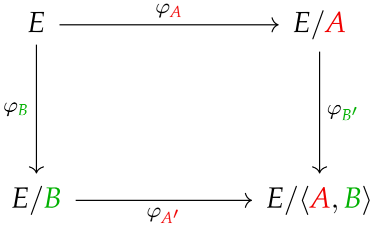

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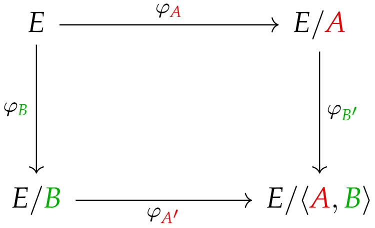

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- They both compute the shared secret

$$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

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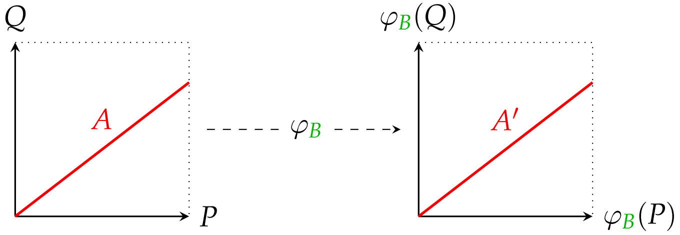

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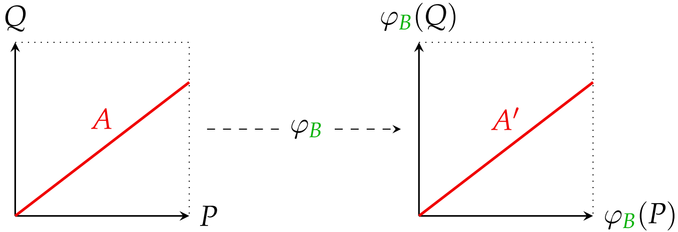

- Alice picks A as $\langle P + [a]Q \rangle$ for fixed public $P, Q \in E$.
- Bob includes $\varphi_B(P)$ and $\varphi_B(Q)$ in his public key.

$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$!

SIDH in one slide

Public parameters:

- a large prime $p = 2^n 3^m - 1$ and a supersingular $E/\mathbb{F}_p$
- bases (P_A, Q_A) and (P_B, Q_B) of $E[2^n]$ and $E[3^m]$

Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \ldots 2^n - 1\}$		$b \xleftarrow{\text{random}} \{0 \ldots 3^m - 1\}$
$A := \langle P_A + [a] Q_A \rangle$ compute $\varphi_A : E \to E/A$		$B := \langle P_B + [b] Q_B \rangle$ compute $\varphi_B : E \to E/B$
$E/A,\ \varphi_A(P_B),\ \varphi_A(Q_B)$	⟶ / ⟵ (exchange)	$E/B,\ \varphi_B(P_A),\ \varphi_B(Q_A)$
$A' := \langle \varphi_B(P_A) + [a] \varphi_B(Q_A) \rangle$ $s := j\big((E/B)/A'\big)$		$B' := \langle \varphi_A(P_B) + [b] \varphi_A(Q_B) \rangle$ $s := j\big((E/A)/B'\big)$

Break it by: given public info, find secret key–φ_A or just A.

Security

Hard Problem:

Given

- supersingular public elliptic curves $E_0/\mathbb{F}_{p^2}$ and $E_A/\mathbb{F}_{p^2}$ connected by a secret 2^n-degree isogeny $\varphi_A : E_0 \to E_A$, and
- the action of φ_A on the 3^m-torsion of E_0,

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- Knowledge of $\mathrm{End}(E_0)$ and $\mathrm{End}(E_A)$ is sufficient to efficiently break it.
- Active attacker can recover secret.
- In SIDH, $\mathrm{End}(E_0)$ is fixed and $3^m \approx 2^n \approx \sqrt{p}$.
- If $3^m > 2^n$ or $3^m, 2^n > \sqrt{p}$, security claims are weakened.

Security of SIKE

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- No commutative group action to exploit here*

What about signatures?

Ex: CSI-FiSh (S '06, D-G '18, Beullens-Kleinjung-Vercauteren '19)

Identification scheme from $H \times S \to S$:

Prover	**Public**	**Verifier**
	$E \in S, \mathfrak{l}_i \in H$	
$s_i \leftarrow \$\, \mathbb{Z}$		
$\mathsf{sk} = \prod \mathfrak{l}_i^{s_i}$,		
$\mathsf{pk} = \mathsf{sk} * E \xrightarrow{\mathsf{pk}}$	pk	
	$\xleftarrow{c}$	$c \leftarrow \$\, \{0,1\}$
$t_i \leftarrow \$\, \mathbb{Z}$		
$\mathsf{esk} = \prod \mathfrak{l}_i^{t_i}$,		
$\mathsf{epk}_1 = \mathsf{esk} * E$,		
$\mathsf{epk}_2 = \mathsf{esk} \cdot \mathsf{sk}^{-c}$	$\xrightarrow{\mathsf{pk}, \mathsf{epk}_1, \mathsf{epk}_2}$	check:
		$\mathsf{epk}_1 = \mathsf{epk}_2 * ([\mathsf{sk}^c] * E)$.

After k challenges c, an imposter succeeds with prob 2^{-k}.

Ex: SQISign (De Feo-Kohel-Leroux-Petit-Wesolowski '20)

Hard Problem in CSIDH, CSI-FiSh, etc:
Given elliptic curves E and $E' \in S$, find $\mathfrak{a} \in H$ such that
$\mathfrak{a} * E = E'$.

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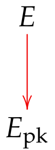

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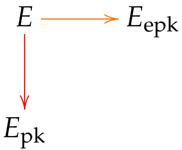

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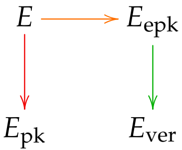

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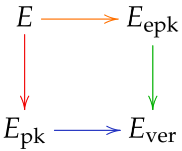

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- SQISign ‘20 Digital signature. Small, slow, clean security assumption, no known attack avenues.

Thank you!

References

[$B^2C^2LMS^2$]	ctidh.isogeny.org
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