How to not break SIDH :(

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CFAIL

Conference for Failed Attempts and Insightful Losses

Submission deadline 2021: 1st May (3-page abstract)

What is this all about?

Diffie–Hellman key exchange '76

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today also elliptic curves)
- an element $g \in G$ of prime order p

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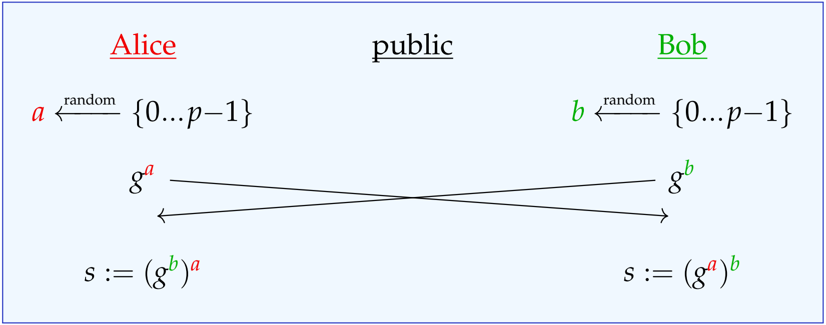

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Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \ldots p-1\}$		$b \xleftarrow{\text{random}} \{0 \ldots p-1\}$
g^a	$\longrightarrow$	g^b
	$\longleftarrow$	
$s := (g^b)^a$		$s := (g^a)^b$

Fundamental reason this works: $\cdot^a$ and $\cdot^b$ are commutative!

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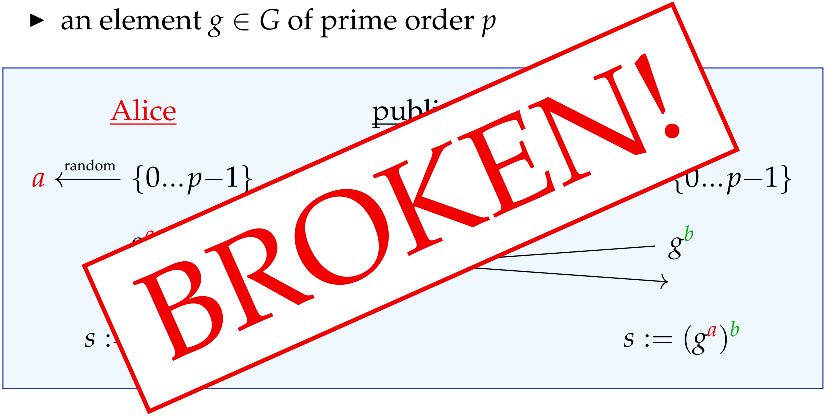

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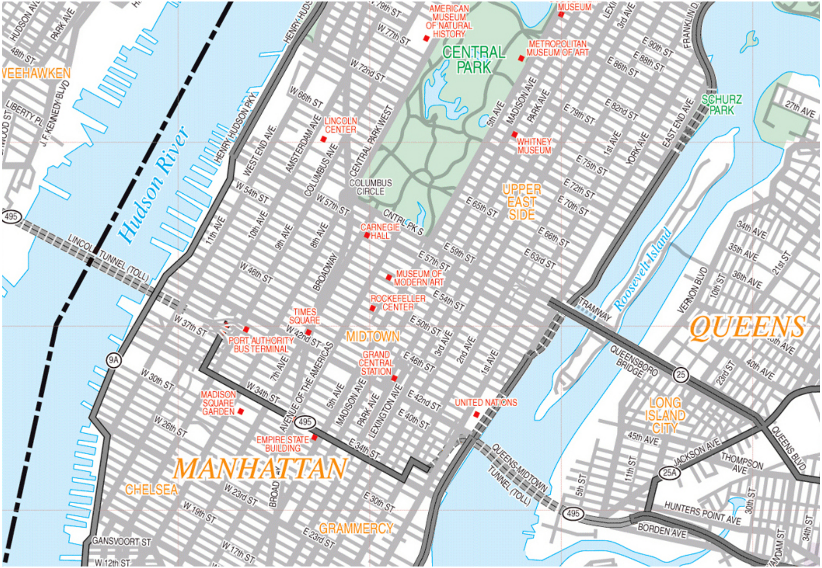
Graph walking Diffie–Hellman?



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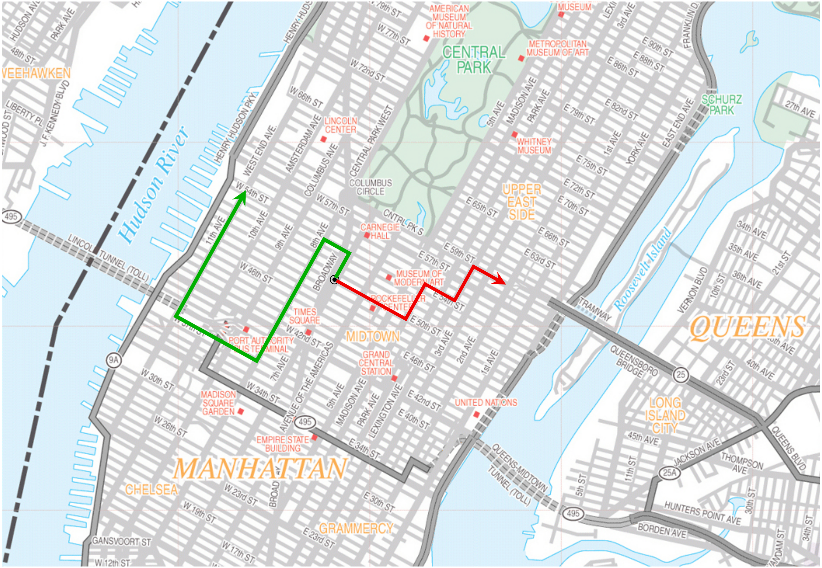

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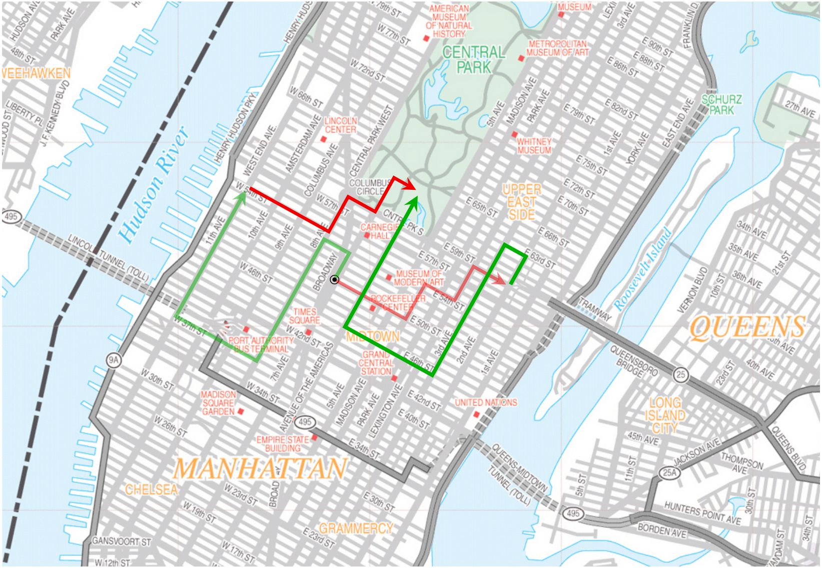

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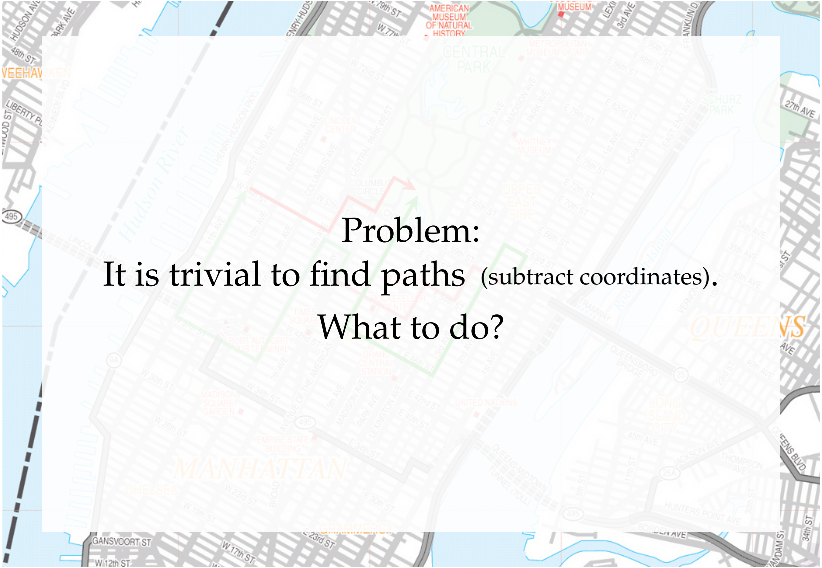

Problem:

It is trivial to find paths (subtract coordinates).

What to do?

Big picture

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It is easy to construct graphs that satisfy *almost* all of these — not enough for crypto!

Stand back!

We're going to do maths.

Maths slide 1/5: Elliptic curves *(nodes)*

An elliptic curve (modulo details) is given by an equation

$$E\colon\ y^2 = x^3 + ax + b.$$

A point on E is a solution to this equation *or* the 'fake' point ∞.

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E is an abelian group: we can 'add' points.

- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

$$\left(\lambda^2 - x_1 - x_2,\ \lambda(2x_1 + x_2 - \lambda^2) - y_1\right)$$

do **not** remember these formulas!

where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + a}{2y_1}$ otherwise.

Maths slide 2/5: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- given by rational functions.
- a group homomorphism.

The degree of a separable* isogeny is the size of its kernel.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m]\colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota\colon (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}\,.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$

defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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The ring of endomorphisms of E is denoted by $\mathrm{End}(E)$.

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Each isogeny $\varphi\colon E \to E'$ has a unique dual isogeny $\widehat{\varphi}\colon E' \to E$ characterized by $\widehat{\varphi} \circ \varphi = \varphi \circ \widehat{\varphi} = [\deg \varphi]$.

Maths slide 3/5: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

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Vélu operates in the field where the points in G live.

$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$

$\rightsquigarrow$ this is why we use supersingular curves!

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- A point $P \in E[m]$ is called an m-torsion point.
- The group $G = \langle P \rangle$ generated by an m-torsion point $P \in E[m]$ is the kernel of an m-isogeny written

$$f : E \to E/G.$$

Maths slide 5/5: Supersingular isogeny graphs

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$E/\overline{\mathbb{F}_p}$ is *supersingular* if it has no points of order p.

Let $S \not\ni p$ denote a set of prime numbers.

For this talk: the supersingular S-isogeny graph consists of:

- vertices given by isomorphism classes of supersingular elliptic curves,
- edges $E — E'$ that represent an ℓ-isogeny $E \to E'$ and its dual $E' \to E$, where $\ell \in S$ (up to isomorphism)

both defined over $\overline{\mathbb{F}_p}$.

SIDH as an isogeny graph

- **Vertices**: isomorphism classes of elliptic curves defined over $\overline{\mathbb{F}_p}$.
- **Edges**: 2- and 3-isogenies of elliptic curves (up to some equivalence).

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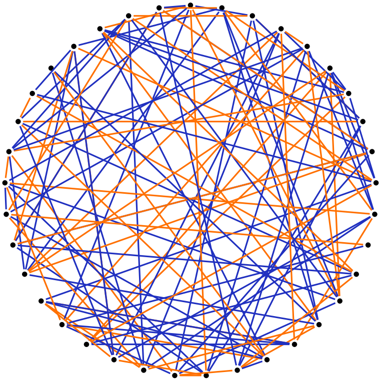

2 and 3-isogenies of elliptic curves over $\mathbb{F}_{431^2}$

Now:

SIDH

SIDH: the dirty details

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Break it by: given public info, find secret key: φ_A (or just A).

Here's some things that don't break it...

Extra points

Aim: given points P_B, Q_B on E, the image E/A of the secret isogeny $\varphi_A : E \to E/A$, and the images $\varphi_A(P_B)$ and $\varphi_B(Q_B)$, find φ_A.

Fact: with the parameters used in SIDH, the images $\varphi_A(P_B)$ and $\varphi_B(Q_B)$ uniquely determine the secret isogeny φ_A.

Extra points: Interpolation?

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  We know enough input-output pairs to determine the map.

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⇝ Rational function interpolation?

☹ ...the polynomials are of exponential degree $\approx \sqrt{p}$.

⇝ can't even write down the result without decomposing into a sequence of smaller-degree maps.

- No known algorithms for interpolating and decomposing at the same time.

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☹ There's an isomorphism of groups

$$E(\mathbb{F}_{p^2}) \cong (\mathbb{Z}/2^n)^2 \times (\mathbb{Z}/3^m)^2 .$$

$\Longrightarrow$ can't learn anything about 2^n from 3^m using groups alone.
(Annoying: This shows up in many disguises.)

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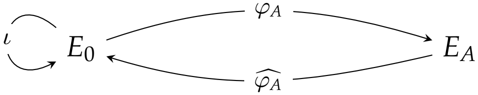

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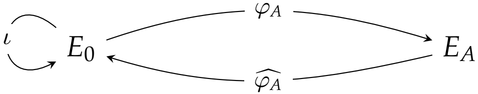

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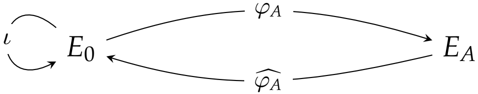

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- Idea: Find an appropriate endomorphism τ of degree $3^m r$; recover 3^m-part as above; brute-force the $r$emaining part.
  $\rightsquigarrow$ image of r-torsion point under φ_A
  $\Longrightarrow$ (details) $\Longrightarrow$ Recover the secret φ_A.

$\frown$ To get r small enough to be an attack, we have to change the SIDH parameters so that Alice's isogeny has a much higher degree than Bob's.

Extra points: Summary

- Same problem all over the place:
  There seems to be no way to obtain *anything* from the given action-on-3^m-torsion except what's given.

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- Life sucks.

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The pure isogeny problem

Fundamental problem: given supersingular E and $E'/\mathbb{F}_{p^2}$ that are ℓ^n-isogeneous, compute an isogeny $\phi : E \to E'$.

The pure isogeny problem

Example

Choose

$$E/\mathbb{F}_{431} : y^2 = x^3 + 1 \quad \text{and} \quad E'/\mathbb{F}_{431} : y^2 = x^3 + 291x + 298.$$

The pure isogeny problem

Example

Choose

$$E/\mathbb{F}_{431} : y^2 = x^3 + 1 \quad \text{and} \quad E'/\mathbb{F}_{431} : y^2 = x^3 + 291x + 298.$$

These elliptic curves are $2^2 = 4$-isogenous. Problem: compute an isogeny $f : E \to E'$.

The kernel of $f : E \to E'$ is generated by a point $P \in E(\overline{\mathbb{F}_p})$ of order 4.

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Solution (b) is meet-in-the-middle: complexity $\tilde{O}(p^{1/4})$.

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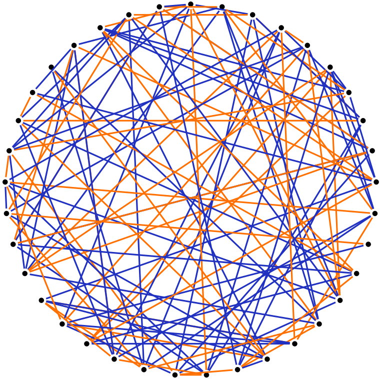

2, 3-isogenies
over $\mathbb{F}_{431^2}$

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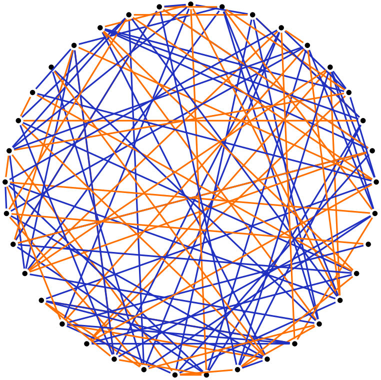

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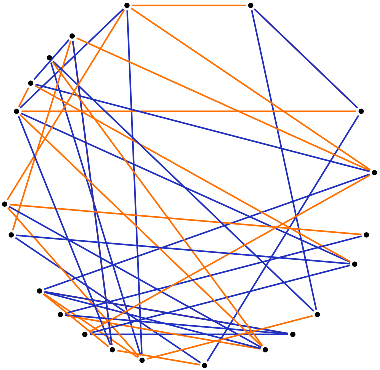

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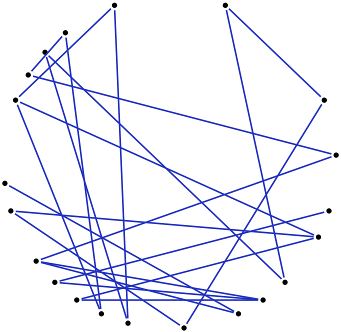

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nodes up to $\overline{\mathbb{F}_{431}}$-isomorphism

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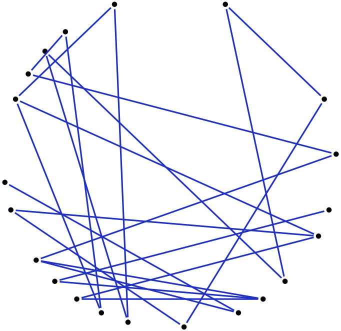

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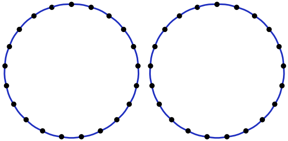

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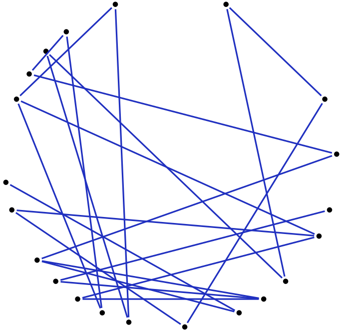

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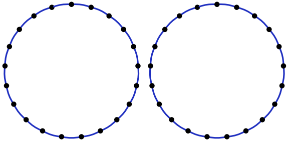

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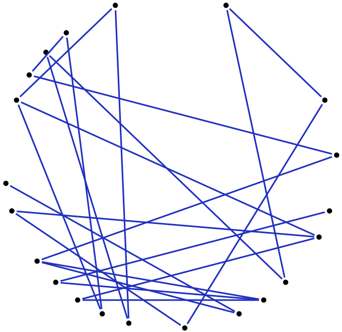

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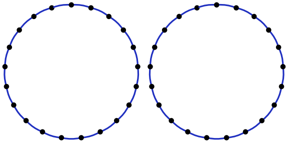

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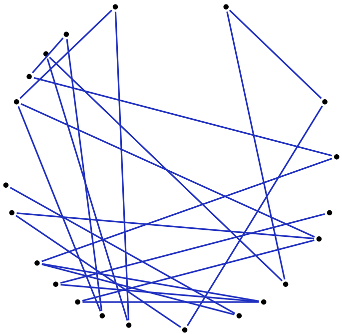

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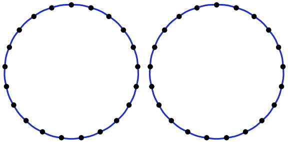

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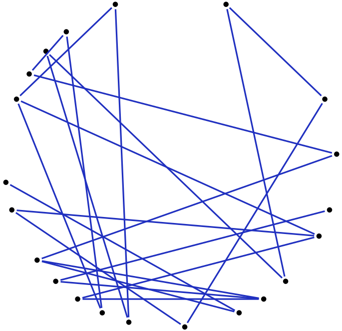

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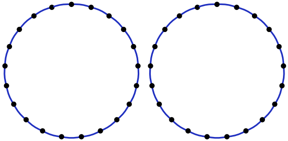

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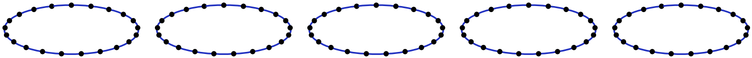
(Delfs-Galbraith, Biasse-Jao-Sankar)

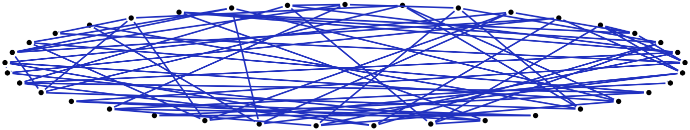
More graphs defined over $\mathbb{F}_p$



From 1-dimensional $E/\mathbb{F}_{p^2}$,

construct 2-dimensional $W(E)/\mathbb{F}_p$

'Weil restriction'



This picture is very unlikely to be accurate.

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More equivalent categories: lifting to $\mathbb{C}$

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Here computing isogenies is easy!

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- If you can find non-scalar endomorphisms, SIDH is probably already broken by earlier work (Kohel-Lauter-Petit-Tignol and Galbraith-Petit-Shani-Ti).

Thank you!