CSIDH:
An Efficient Post-Quantum Commutative Group Action

https://csidh.isogeny.org

Wouter Castryck[1] Tanja Lange[2] Chloe Martindale[2]
Lorenz Panny[2] Joost Renes[3]

[1]KU Leuven [2]TU Eindhoven [3]RU Nijmegen

Cryptography Seminar, Rennes, France, 1st February 2019

[ˈsiːˌsaɪd]

History

1976 Diffie-Hellman: Key exchange using exponentiation in groups (DH)

1985 Koblitz-Miller: Diffie-Hellman style key exchange using multiplication in elliptic curve groups (ECDH)

1990 Brassard-Yung: Generalizes 'group exponentiation' to 'groups acting on sets' in a crypto context

1994 Shor: Polynomial-time quantum algorithm to break the discrete logarithm problem in any group, quantumly breaking DH and ECDH

1997 Couveignes: Post-quantum isogeny-based Diffie-Hellman-style key exchange using commutative group actions (not published at the time)

2003 Kuperberg: Subexponential-time quantum algorithm to attack cryptosystems based on a hidden shift

History

2004 Stolbunov-Rostovtsev independently rediscover Couveignes' scheme (CRS)

2006 Charles-Goren-Lauter: Build hash function from supersingular isogeny graph

2010 Childs-Jao-Soukharev: Apply Kuperberg's (and Regev's) hidden shift subexponential quantum algorithm to CRS

2011 Jao-De Feo: Build Diffie-Hellman style key exchange from supersingular isogeny graph (SIDH)

2018 De Feo-Kieffer-Smith: Apply new ideas to speed up CRS

2018 Castryck-Lange-Martindale-Panny-Renes: Apply ideas of De Feo, Kieffer, Smith to supersingular curves over $\mathbb{F}_p$ (CSIDH)

(History slides mostly stolen from Wouter Castryck)

Why CSIDH?

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- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim 35\,\mathrm{ms}$ per operation
- Flexible:
  - [DG] uses CSIDH for 'SeaSign' signatures
  - [DGOPS] uses CSIDH for oblivious transfer
  - [FTY] uses CSIDH for authenticated group key exchange

Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group G via the map

$$\begin{array}{rcl} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x. \end{array}$$

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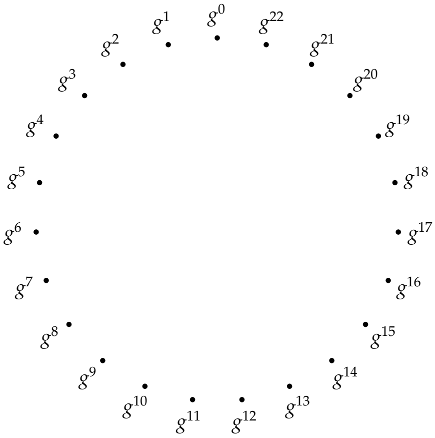
$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Square-and-multiply

Suppose $\#G = 23$ and that Alice computes g^{13}.



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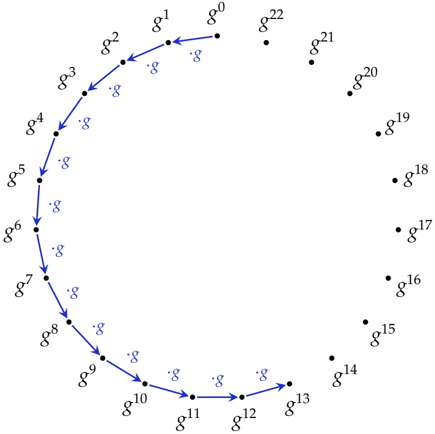

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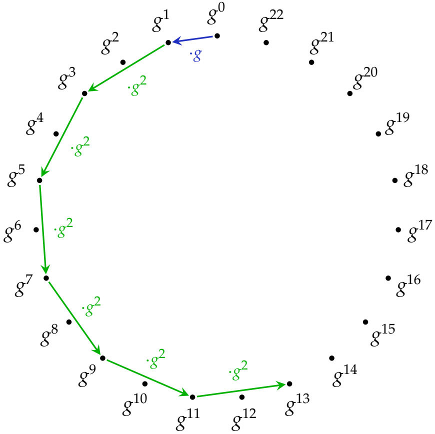

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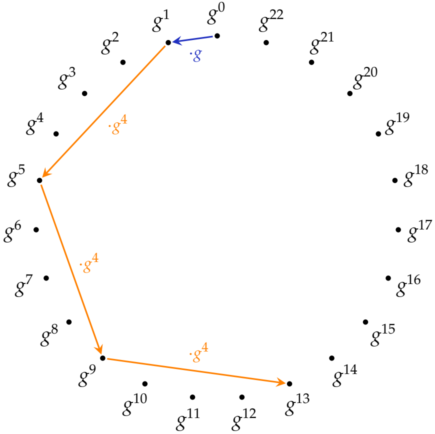

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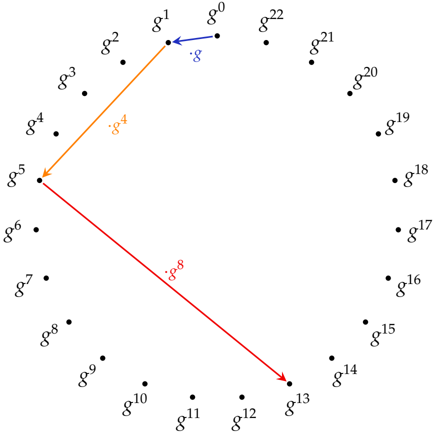

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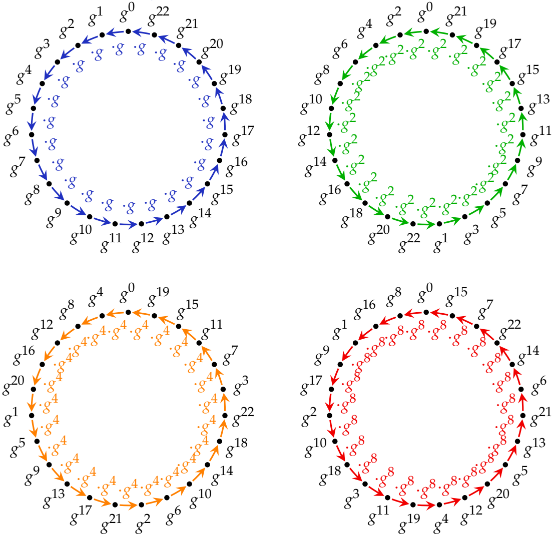

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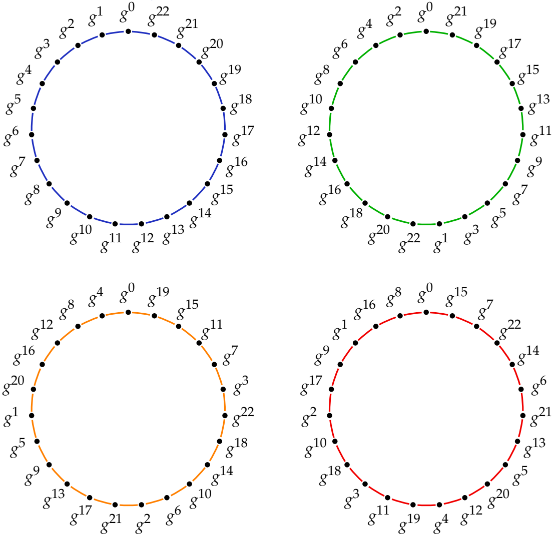

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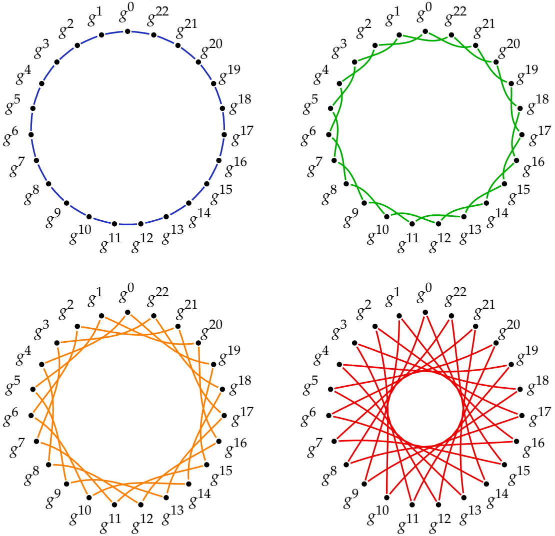

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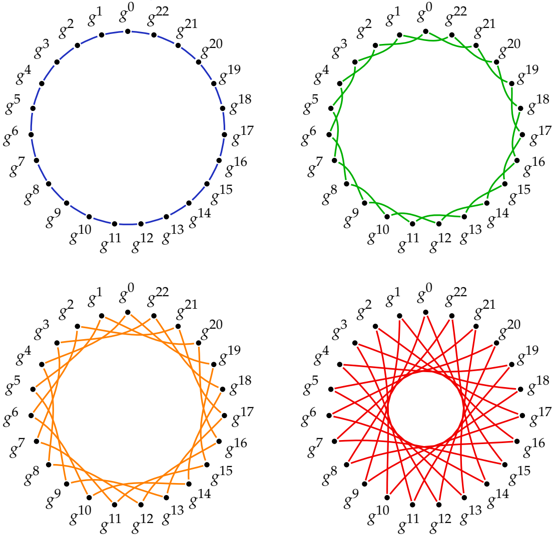
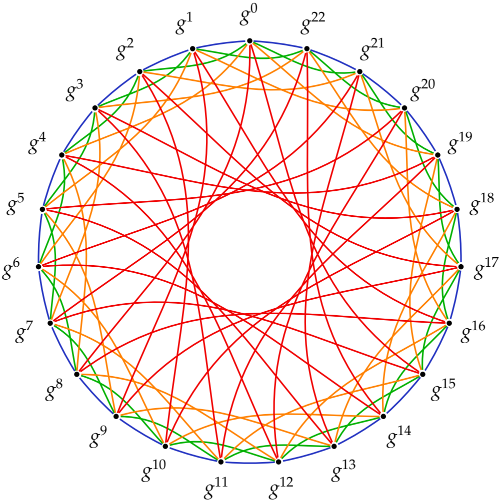


Cycles are compatible: [right, then left] = [left, then right], etc.

Union of cycles: rapid mixing



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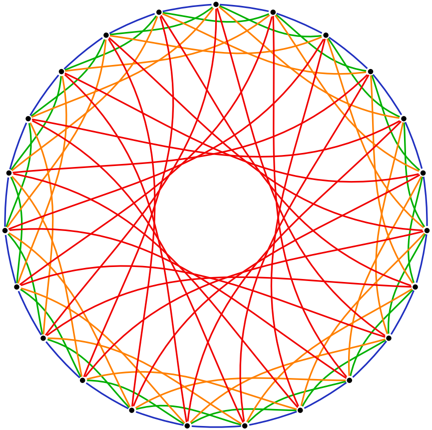
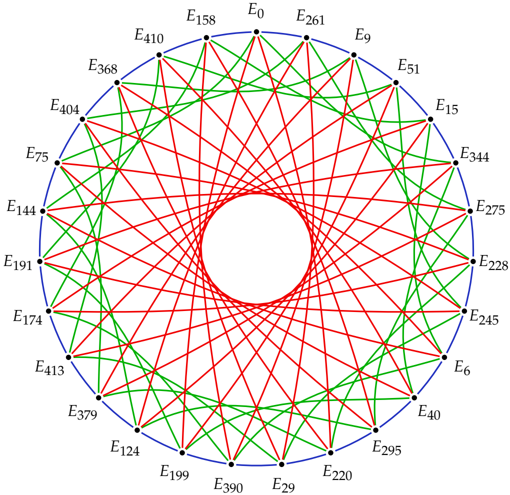


CSIDH: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



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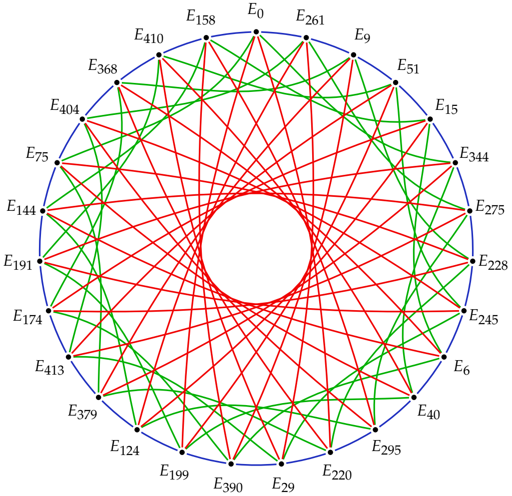

Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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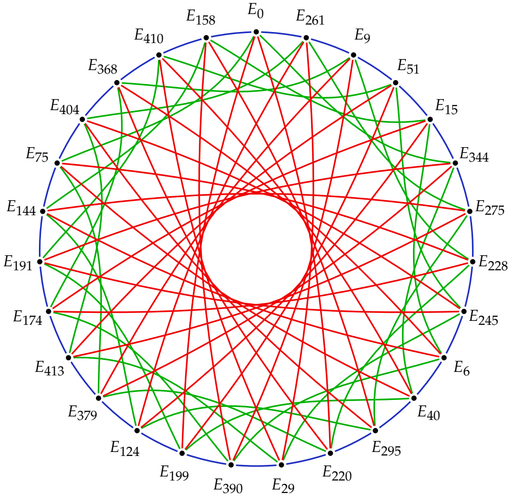

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Edges: 3-, 5-, and 7-isogenies.

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- The set of $\mathbb{F}_p$-rational solutions (x, y) to an elliptic curve equation $E_A/\mathbb{F}_p$, together with a 'point at infinity' P_∞, forms a group with identity P_∞, notated $E_A(\mathbb{F}_p)$.

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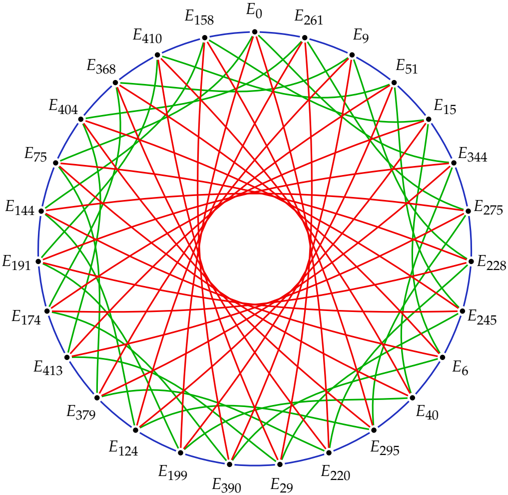
- Every ℓ-isogeny $f : E_A \to E_B$ has a unique dual ℓ-isogeny $\hat{f} : E_B \to E_A$. $\rightsquigarrow$ Undirected edges!

Graphs of elliptic curves



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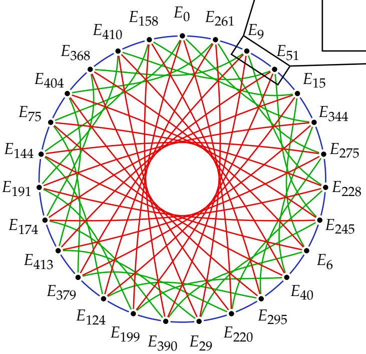
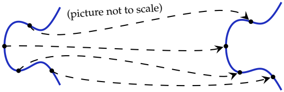


A 3-isogeny



$$E_{51}\colon y^2 = x^3 + 51x^2 + x \longrightarrow E_9\colon y^2 = x^3 + 9x^2 + x$$

$$(x, y) \longmapsto \left(\frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97},\ y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$$

(...and its dual isogeny)

Quantumifying Exponentiation

- Recall: we want to replace the exponentiation map

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- Replace $\mathbb{Z}$ by a commutative group H... more details to come!
- The action of a well-chosen h (or h^{-1}) $\in H$ on $E_A \in S$ gives an elliptic curve one step from E_A around one of the cycles in a $+$ (or $-$) direction.

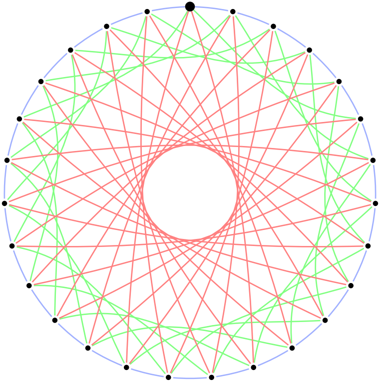
Diffie-Hellman on isogeny graphs

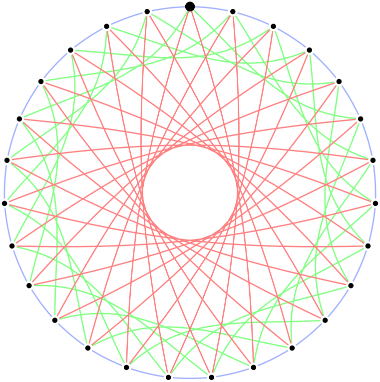
Alice

$[+, -, +, -]$

Bob

$[+, +, -, +]$





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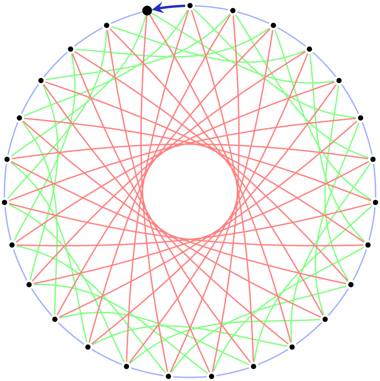

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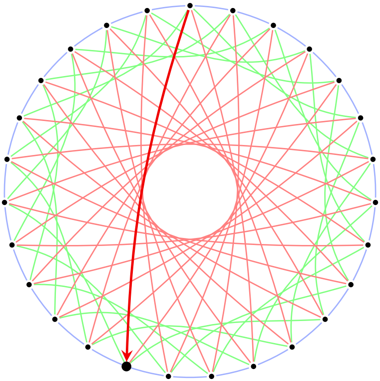

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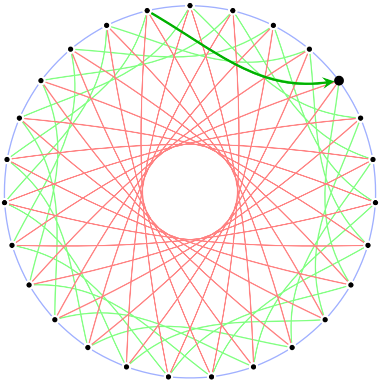

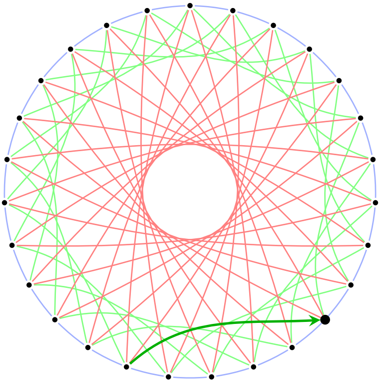

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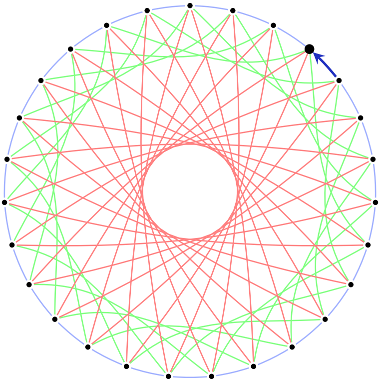

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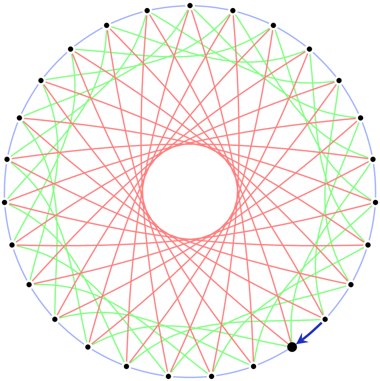

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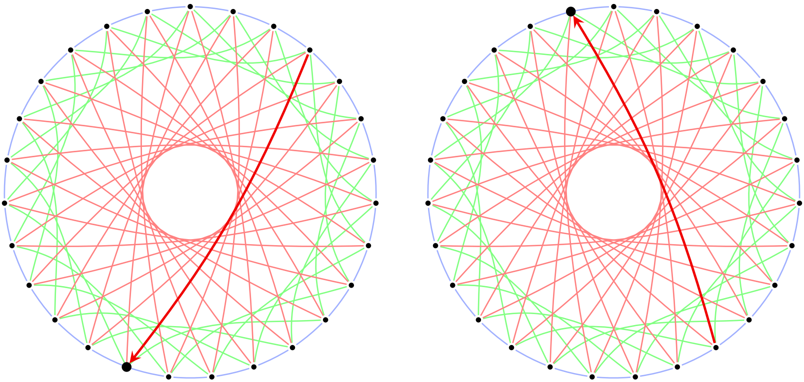

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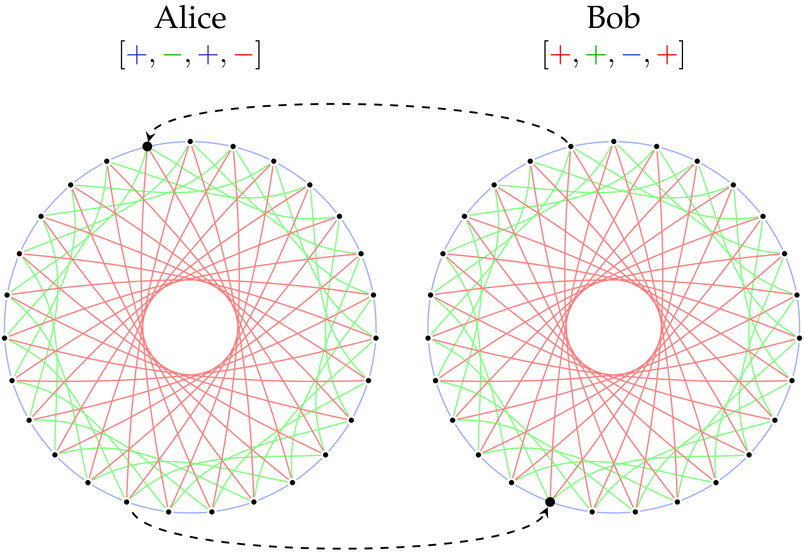

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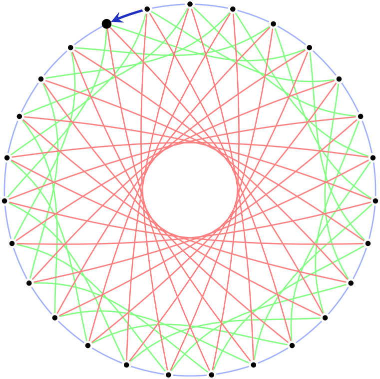

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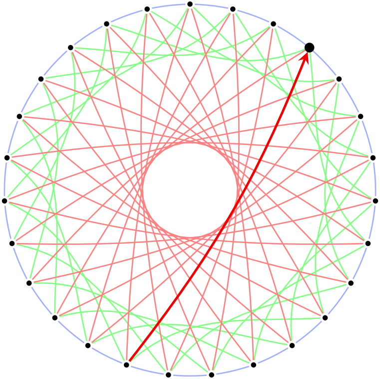

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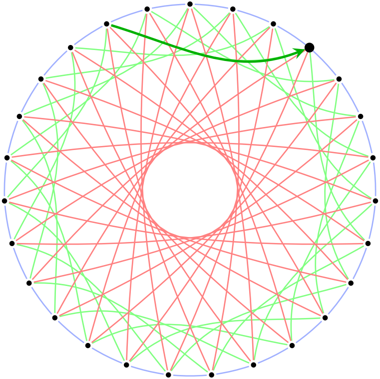

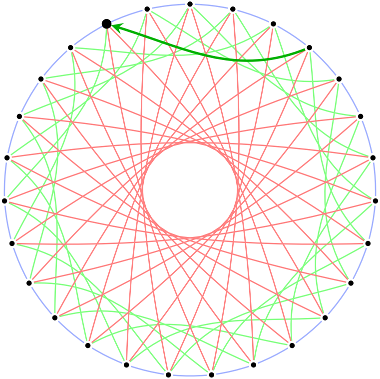

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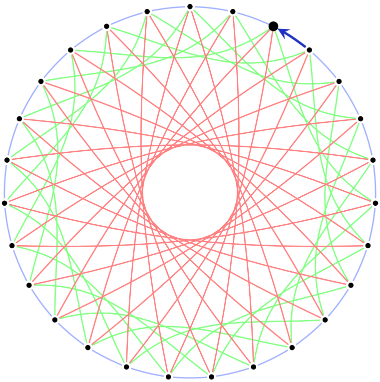

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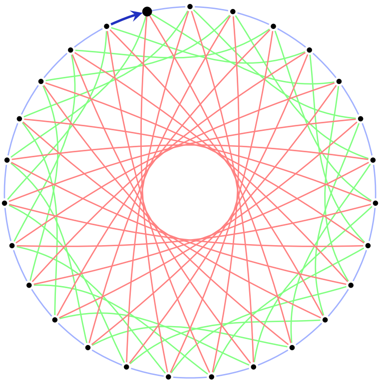

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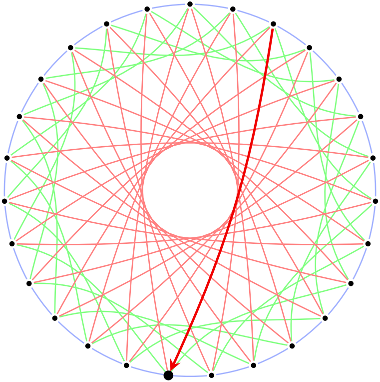

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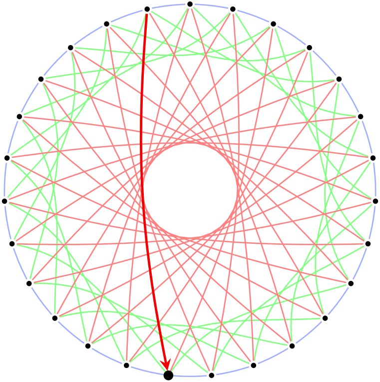

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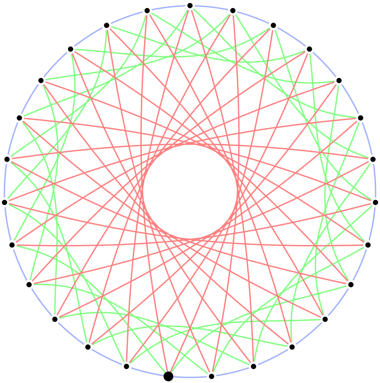

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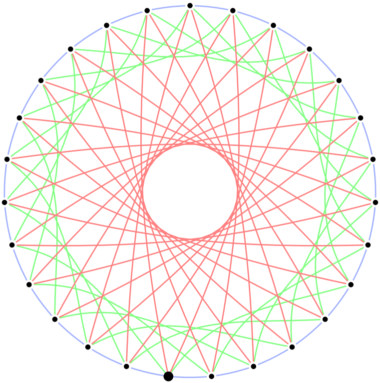

A walkable graph

- Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
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Important properties for such a walk:

IP1 ▸ The graph is a composition of compatible cycles.

IP2 ▸ We can compute neighbours in given directions.

Towards IP1: Isogeny graphs

Definition

Let p and ℓ be distinct primes. The isogeny graph G_ℓ containing $E/\mathbb{F}_p$ is the graph with:

- Nodes: elliptic curves $E'/\mathbb{F}_p$ (up to $\mathbb{F}_p$-isomorphism) with $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$.
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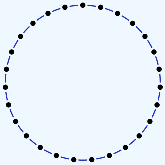
G_3: 

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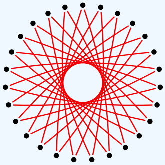
G_5: 

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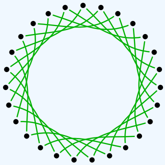

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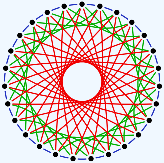
$G_3 \cup G_5 \cup G_7$: 

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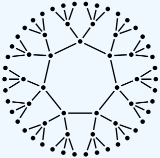
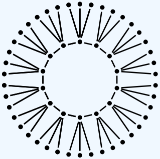
G_3:  G_5: 

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- Two nodes are at different distances from the cycle if and only if they have different endomorphism rings.

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The $\mathbb{F}_p$-rational endomorphism ring $\mathrm{End}_{\mathbb{F}_p}(E)$ of an elliptic curve $E/\mathbb{F}_p$ is the set of $\mathbb{F}_p$-rational endomorphisms.

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Example

Let $p \geq 5$, let $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ be a supersingular elliptic curve, and let π be the Frobenius endomorphism. Then

$$\pi \circ \pi = [-p]$$

and

$$\begin{array}{ccc} \mathbb{Z}[\sqrt{-p}] & \to & \mathrm{End}_{\mathbb{F}_p}(E) \\ n & \mapsto & [n] \\ \sqrt{-p} & \mapsto & \pi \end{array}$$

extends $\mathbb{Z}$-linearly to a ring homomorphism.

Towards IP1: Group action

For $p \equiv 3 \pmod 8$ and $p \geq 5$, if $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ is supersingular, then $\mathrm{End}_{\mathbb{F}_p}(E_A) \cong \mathbb{Z}[\sqrt{-p}]$.

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- What is the action?

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- For $[I] \in \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}])$, let $\tilde{I}$ be an integral representative of the ideal class $[I]$. Then
$$\begin{array}{ccc} \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}]) \times S & \to & S \\ ([I], E_A) & \mapsto & f_{\tilde{I}}(E_A) \end{array}$$
is a free, transitive group action!

IP1: The graph is a composition of compatible cycles

- The nodes of graph G_ℓ are the elements of the set S of supersingular elliptic curves $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ with $p \equiv 3 \pmod 8$ and $p \geq 5$.

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$\rightsquigarrow$ there is a choice of $\ell_1, \ldots, \ell_n$ such that $G_{\ell_1} \cup \cdots \cup G_{\ell_n}$ is a composition of compatible cycles (IP1).

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- Choosing the direction in the graph corresponds to choosing this sign.

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To compute a neighbour of E_A, we have to compute an ℓ-isogeny from E_A. To do this:

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  - Given a $\mathbb{F}_p$-rational point of order ℓ, the isogeny computations can be done over $\mathbb{F}_p$.

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Given the group action as above, Vélu's formulas give actual isogenies!
With our design choices all isogeny computations are over $\mathbb{F}_p$. [1]

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$\Rightarrow$ Tiny keys!

Does any A work?

[2] This algorithm has a small chance of false positives, but we actually use a variant that *proves* that E_A has $p + 1$ points.

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No.

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p + 1$ points.
  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p + 1]P = \infty$.[2]

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Classical Security

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- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from E_0 to E_A, whereas an attacker has compute all the possible paths from E_0.
- Best classical attacks are (variants of) meet-in-the-middle: Time $O(\sqrt[4]{p})$.

Quantum Security

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- Childs-Jao-Soukharev [CJS] applied Kuperberg/Regev to CRS – their attack also applies to CSIDH.
- Part of CJS attack computes many paths in superposition.

Quantum Security

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
  - Choice of time/memory trade-off (Regev/Kuperberg)
  - Quantum evaluation of isogenies

  (and much more).

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- For fastest variant of Kuperberg (uses billions of qubits), total cost of CSIDH-512 attack is about 2^{81} qubit operations.[3]

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Parameters

CSIDH-log p	intended NIST level	public key size	private key size	time (full exchange)	cycles (full exchange)	stack memory	classical security
CSIDH-512	1	64 b	32 b	70 ms	212e6	4368 b	128
CSIDH-1024	3	128 b	64 b				256
CSIDH-1792	5	224 b	112 b				448

CSIDH vs SIDH?

Apart from mathematical background, SIDH and CSIDH actually have very little in common, and are likely to be useful for different applications.

Here is a comparison for (conjectured) NIST level 1:

	CSIDH	SIDH
Speed (NIST 1)	70ms (can be improved)	$\approx$ 10ms[4]
Public key size (NIST 1)	64B	378B
Key compression (speed)		$\approx$ 15ms
Key compression (size)		222B
Constant-time slowdown	$\approx \times 3$ (can be improved)	$\approx \times 1$
Submitted to NIST	no	yes
Maturity	9 months	8 years
Best classical attack	$p^{1/4}$	$p^{1/4}$
Best quantum attack	$L_p[1/2]$	$p^{1/6}$
Key size scales	quadratically	linearly
Security assumption	isogeny walk problem	ad hoc
Non-interactive key exchange	yes	unbearably slow
Signatures (classical)	unbearably slow	seconds
Signatures (quantum)	seconds	still seconds?

[4] This is a very conservative estimate!

Work in progress & future work

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- Hardware implementation.
- More applications.
- [Your paper here!]


Thank you!

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Mentioned in this talk:

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Credits: thanks to Lorenz Panny for many of these slides, including all of the beautiful tikz pictures.