CSIDH:
An Efficient Post-Quantum Commutative Group Action

https://csidh.isogeny.org

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[ˈsiːˌsaɪd]

History

- 1976 Diffie-Hellman: Key exchange using exponentiation in groups (DH)
- 1985 Koblitz-Miller: Diffie-Hellman style key exchange using multiplication in elliptic curve groups (ECDH)
- 1990 Brassard-Yung: Generalizes 'group exponentiation' to 'groups acting on sets' in a crypto context
- 1994 Shor: Polynomial-time quantum algorithm to break the discrete logarithm problem in any group, quantumly breaking DH and ECDH
- 1997 Couveignes: Post-quantum isogeny-based Diffie-Hellman-style key exchange using commutative group actions (not published at the time)
- 2003 Kuperberg: Subexponential-time quantum algorithm to attack cryptosystems based on a hidden shift

History

2004 Stolbunov-Rostovtsev independently rediscover Couveignes' scheme (CRS)

2006 Charles-Goren-Lauter: Build hash function from supersingular isogeny graph

2010 Childs-Jao-Soukharev: Apply Kuperberg's (and Regev's) hidden shift subexponential quantum algorithm to CRS

2011 Jao-De Feo: Build Diffie-Hellman style key exchange from supersingular isogeny graph (SIDH)

2018 De Feo-Kieffer-Smith: Apply new ideas to speed up CRS

2018 Castryck-Lange-Martindale-Panny-Renes: Apply ideas of De Feo, Kieffer, Smith to supersingular curves over $\mathbb{F}_p$ (CSIDH)

(History slides mostly stolen from Wouter Castryck)

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- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim$ 35 ms per operation
- Flexible:
  - [DG] uses CSIDH for 'SeaSign' signatures
  - [DGOPS] uses CSIDH for oblivious transfer
  - [FTY] uses CSIDH for authenticated group key exchange

Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group G via the map

$$\begin{array}{rcl} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x. \end{array}$$

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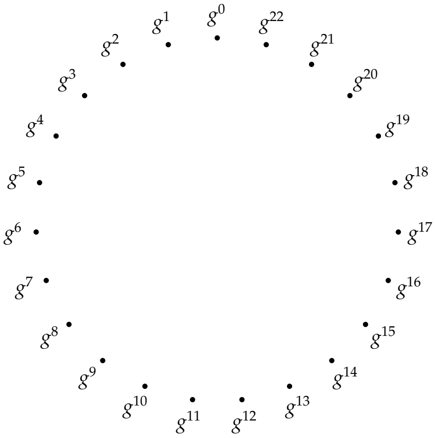
$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Square-and-multiply

Suppose $\#G = 23$ and that Alice computes g^{13}.



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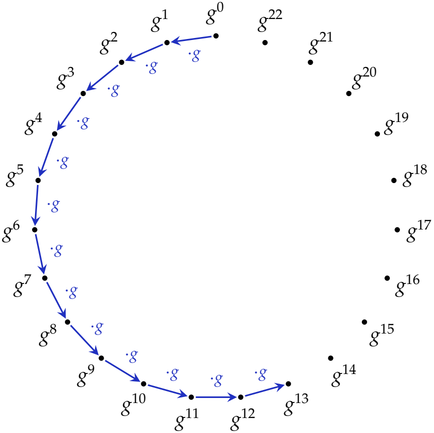

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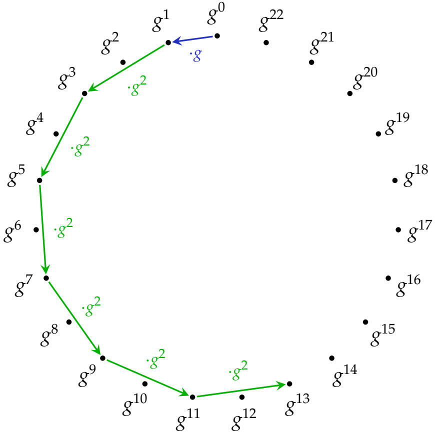

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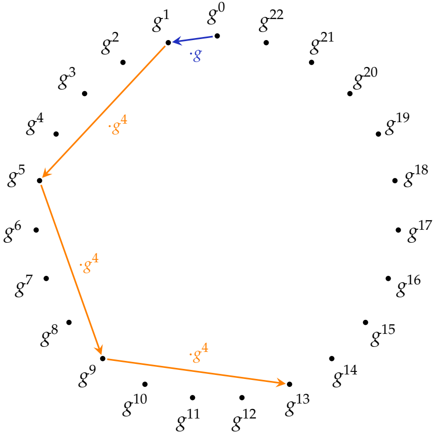

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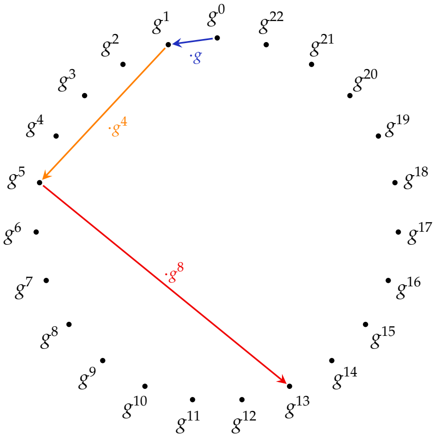

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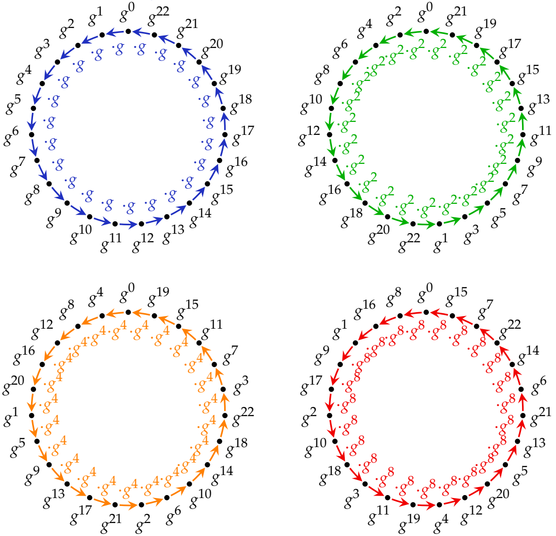

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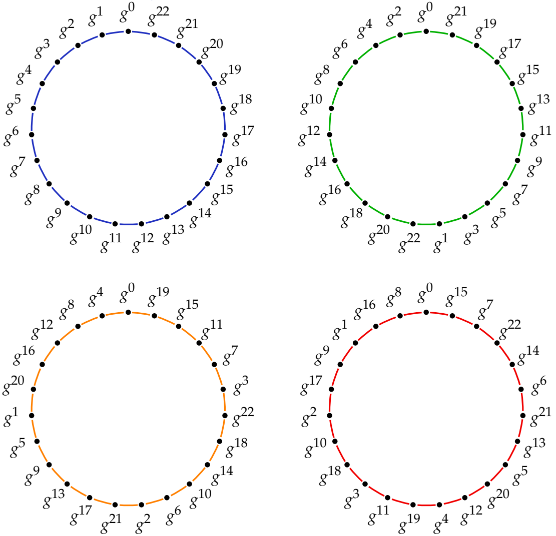

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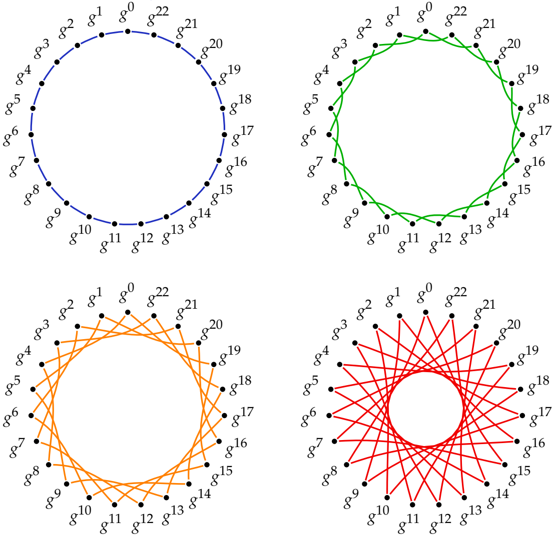

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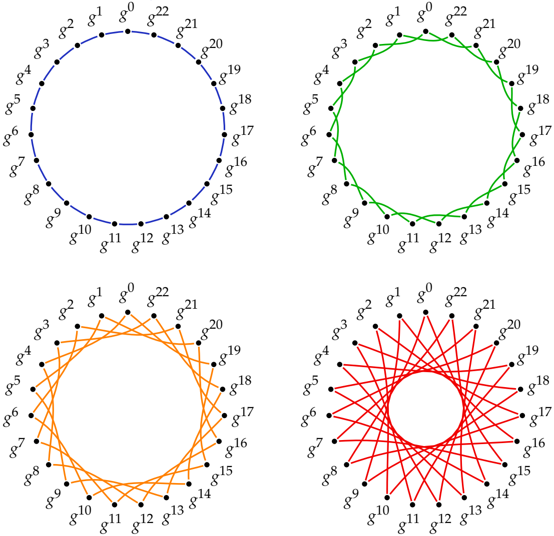
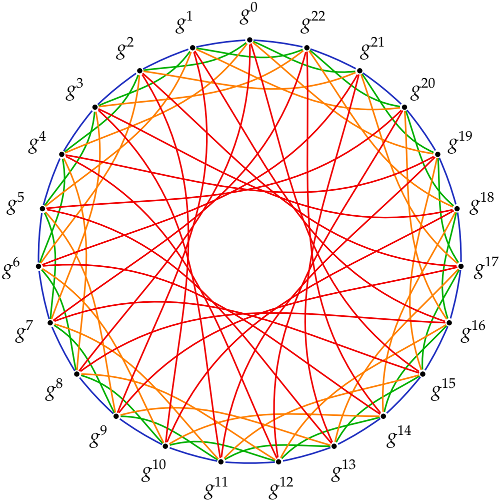


Cycles are compatible: [right, then left] = [left, then right], etc.

Union of cycles: rapid mixing



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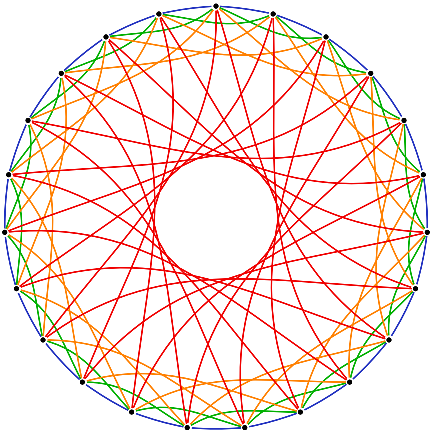
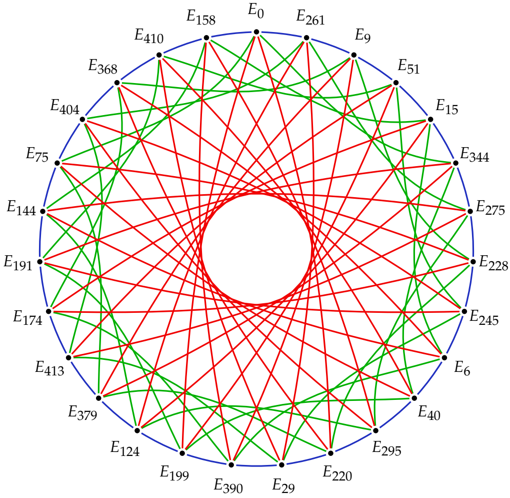


CSIDH: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



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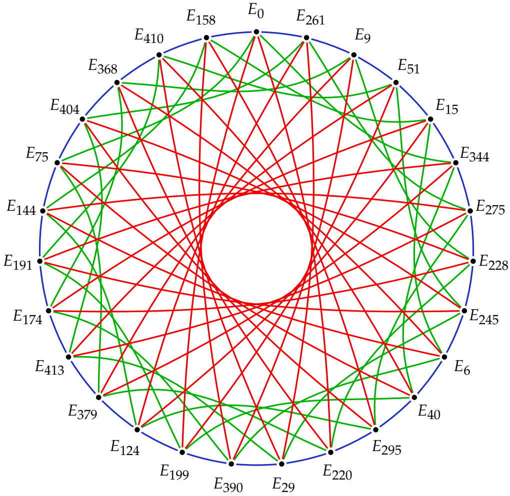

Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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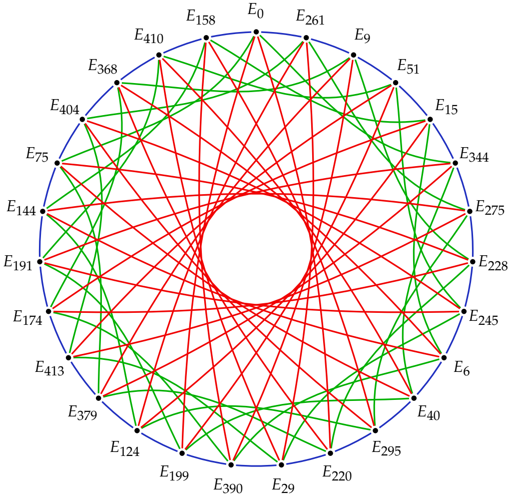

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Edges: 3-, 5-, and 7-isogenies.

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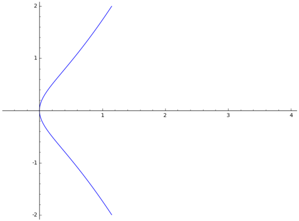

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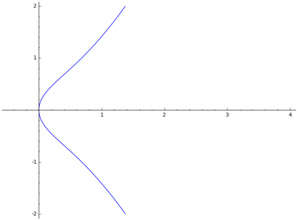

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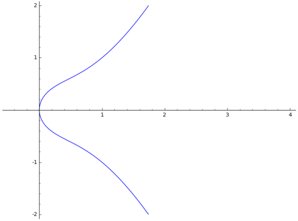

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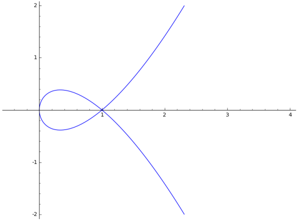

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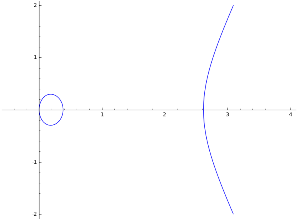

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Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

- If equation E_A is smooth (no self intersections or cusps) it represents an elliptic curve.
- The set of $\mathbb{F}_p$-rational solutions (x, y) to an elliptic curve equation $E_A/\mathbb{F}_p$, together with a 'point at infinity' P_∞, forms a group with identity P_∞, notated $E_A(\mathbb{F}_p)$.

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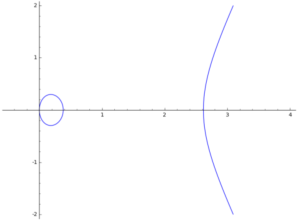

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- An elliptic curve $E_A/\mathbb{F}_p$ with $p \geq 5$ such that $\#E_A(\mathbb{F}_p) = p + 1$ is supersingular.

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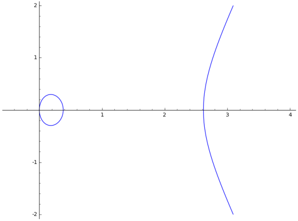

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Edges: 3-, 5-, and 7-isogenies.

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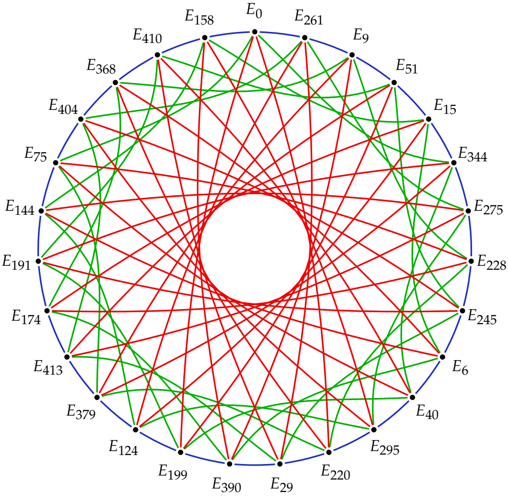
- Every ℓ-isogeny $f : E_A \to E_B$ has a unique dual ℓ-isogeny $f : E_B \to E_A$.

Graphs of elliptic curves



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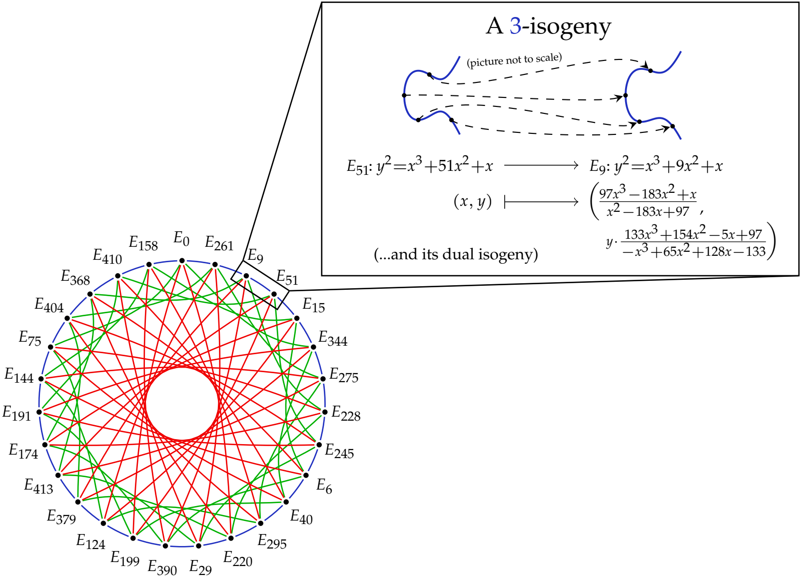
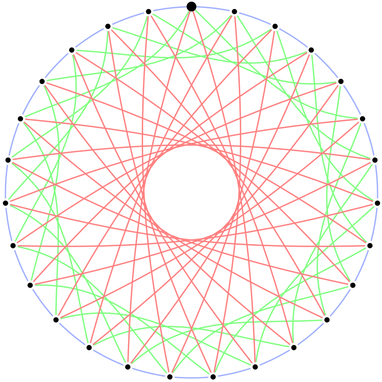


Diffie-Hellman on isogeny graphs

Alice

[+, −, +, −]



Bob

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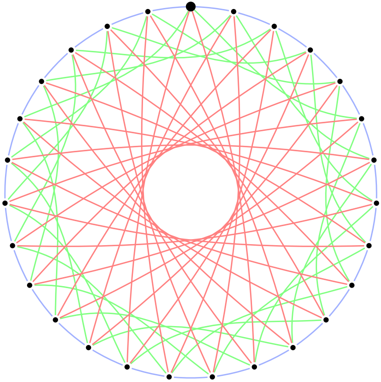

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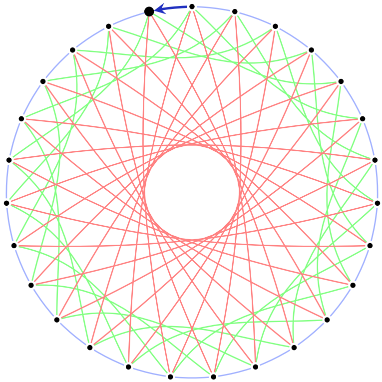

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$\uparrow$

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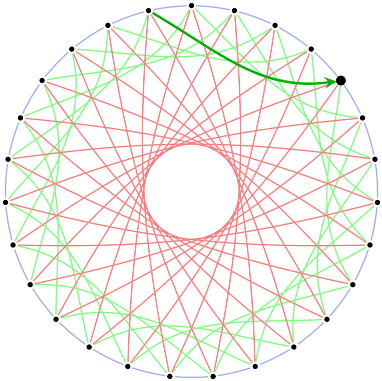

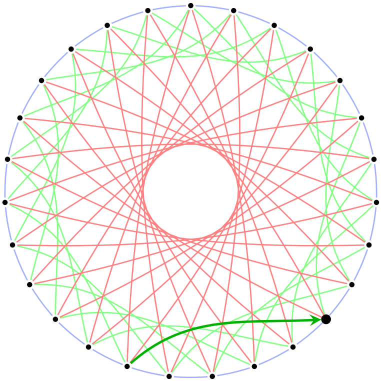

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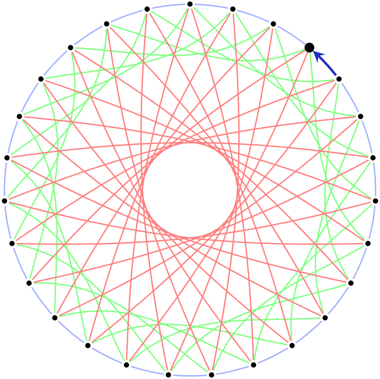

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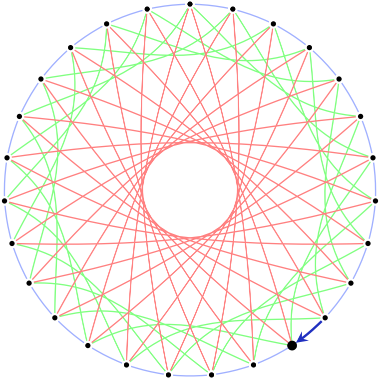

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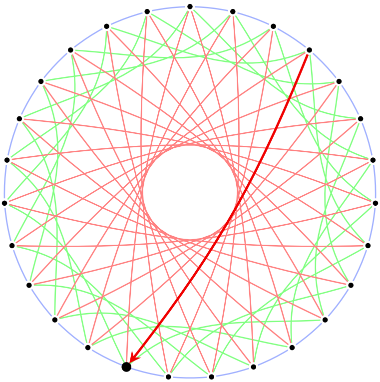

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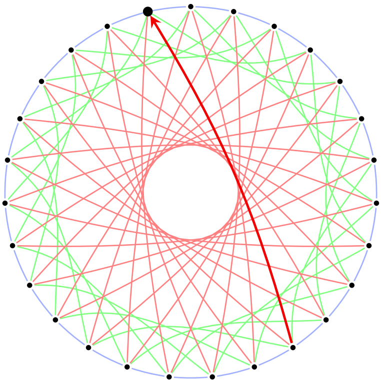

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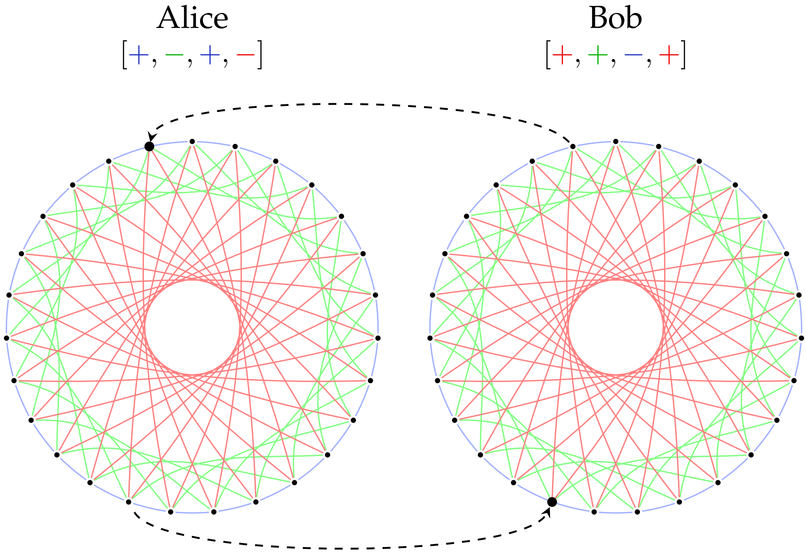

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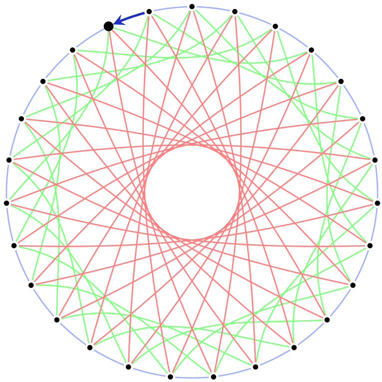

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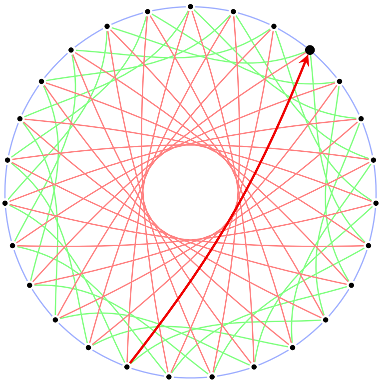

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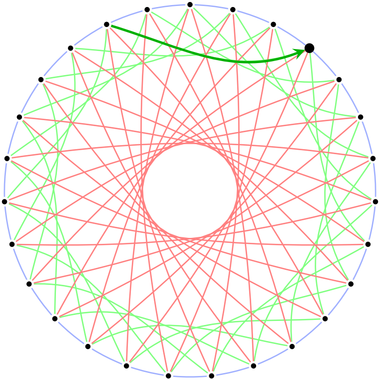

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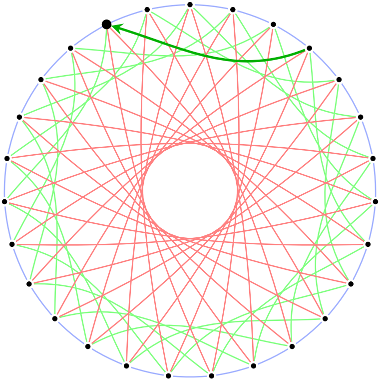

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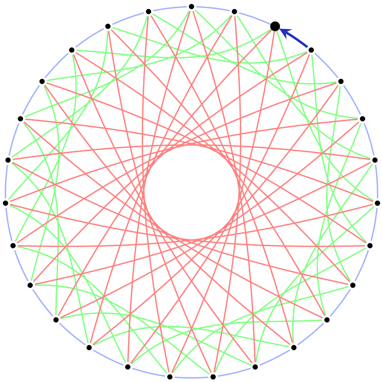

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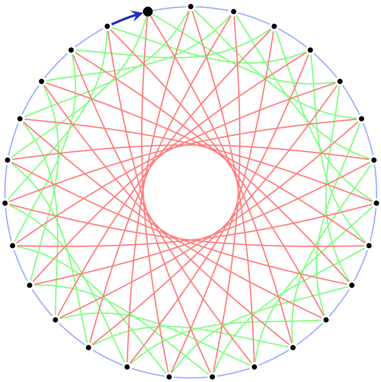

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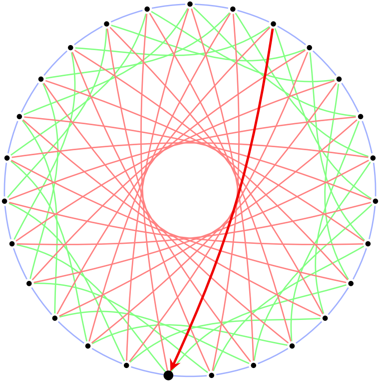

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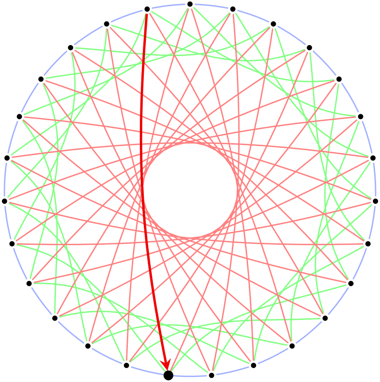

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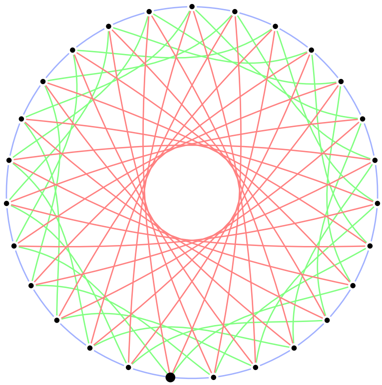

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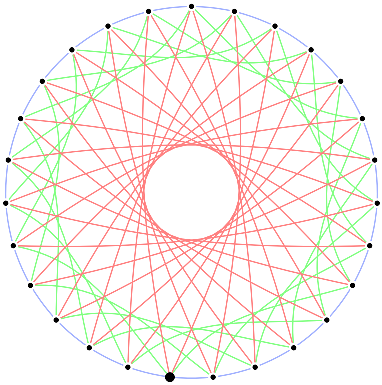

A walkable graph

Important properties for our graph:

IP1 ▸ The graph is a composition of compatible cycles.

IP2 ▸ We can compute neighbours in given directions.

IP1: A composition of cycles

- The graph used in CSIDH is constructed as a composition of graphs G_ℓ of ℓ-isogenies.

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- In our example, these are

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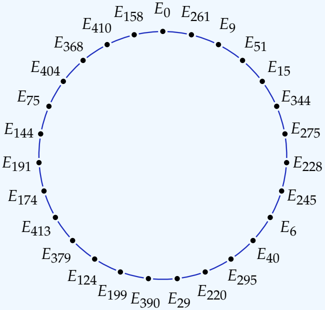

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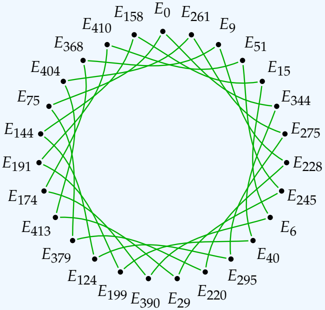
G_5:



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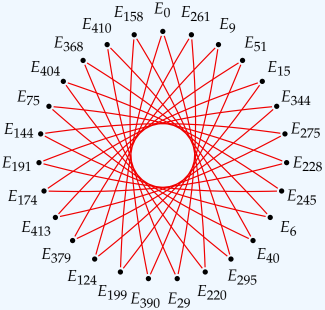
G_7:



IP1: A composition of cycles

- The graph used in CSIDH is constructed as a composition of graphs G_ℓ of ℓ-isogenies.

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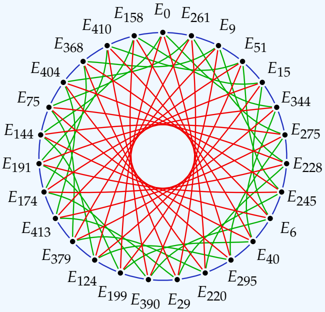
$G_3 \cup G_5 \cup G_7$:



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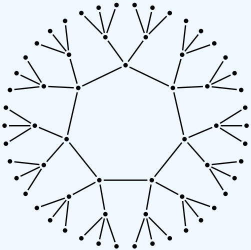
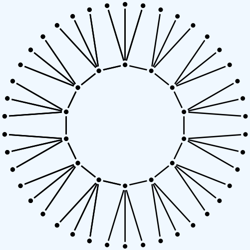
G_3:  G_5: 

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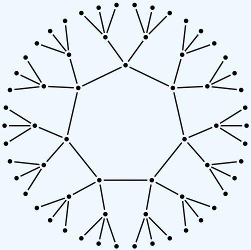
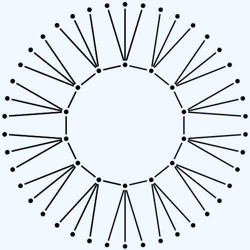
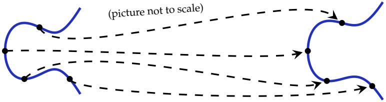
G_3: 

G_5: 

- We want to make sure G_ℓ is just a cycle.

IP2: Compute neighbours in given directions

The edges of G_ℓ are ℓ-isogenies.



$$\begin{array}{rcl} E_{51} : y^2 = x^3 + 51x^2 + x & \longrightarrow & E_9 : y^2 = x^3 + 9x^2 + x \\ (x, y) & \longmapsto & \left(\frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97}, y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right) \end{array}$$

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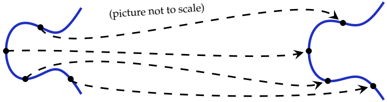

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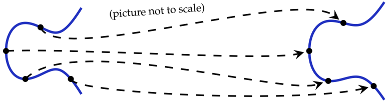

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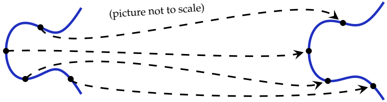

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- Generally needs big extension fields...

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- All G_{ℓ_i} are compatible.
- Computations need only $\mathbb{F}_p$-arithmetic (because $\ell_i | (p + 1)$).

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$\Rightarrow$ Tiny keys!

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p+1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p+1]P = \infty$.[1]

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Classical Security

- Security is based on the isogeny problem: given two elliptic curves, compute an isogeny between them.

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- Best classical attacks are (variants of) meet-in-the-middle: Time $O(\sqrt[4]{p})$.

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- Part of CJS attack computes many paths in superposition.

Quantum Security

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
  - Choice of time/memory trade-off (Regev/Kuperberg)
  - Quantum evaluation of isogenies

  (and much more).

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- For fastest variant of Kuperberg (uses billions of qubits), total cost of CSIDH-512 attack is about 2^{81} qubit operations.[2]

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Parameters

CSIDH-log p	intended NIST level	public key size	private key size	time (full exchange)	cycles (full exchange)	stack memory	classical security
CSIDH-512	1	64 b	32 b	65 ms	212e6	4368 b	128
CSIDH-1024	3	128 b	64 b				256
CSIDH-1792	5	224 b	112 b				448

CSIDH vs SIDH?

Apart from mathematical background, SIDH and CSIDH actually have very little in common, and are likely to be useful for different applications.

Here is a comparison for (conjectured) NIST level 1:

	CSIDH	SIDH
Speed (NIST 1)	65ms (can be improved)	$\approx$ 10ms[3]
Public key size (NIST 1)	64B	378B
Key compression (speed)		$\approx$ 15ms
Key compression (size)		222B
Constant-time slowdown	$\approx \times 2.2$ (can be improved)	$\approx \times 1$
Submitted to NIST	no	yes
Maturity	1 year	8 years
Best classical attack	$p^{1/4}$	$p^{1/4}$
Best quantum attack	$L_p[1/2]$	$p^{1/6}$
Key size scales	quadratically	linearly
Security assumption	isogeny walk problem	ad hoc
Non-interactive key exchange	yes	unbearably slow
Signatures (classical)	unbearably slow[4]	seconds
Signatures (quantum)	seconds	still seconds?

[3] This is a very conservative estimate!

[4] Word on the street: soon to be milliseconds!

Work in progress & future work

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- [Your paper here!]


Thank you!

References

Mentioned in this talk:

BLMP Bernstein, Lange, Martindale, and Panny:
Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies
`https://quantum.isogeny.org`

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Quantum Security Analysis of CSIDH and Ordinary Isogeny-based Schemes
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CJS Childs, Jao, and Soukharev:
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DG De Feo, Galbraith:
SeaSign: Compact isogeny signatures from class group actions
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DKS De Feo, Kieffer, Smith:
Towards practical key exchange from ordinary isogeny graphs
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Secure Oblivious Transfer from Semi-Commutative Masking
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FTY Fujioka, Takashima, and Yoneyama:
One-Round Authenticated Group Key Exchange from Isogenies
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MCR Meyer, Campos, Reith:
On Lions and Elligators: An efficient constant-time implementation of CSIDH
`https://eprint.iacr.org/2018/1198`

Kup1 Kuperberg:
A subexponential-time quantum algorithm for the dihedral hidden subgroup problem
`https://arxiv.org/abs/quant-ph/0302112`

Kup2 Kuperberg:
Another subexponential-time quantum algorithm for the dihedral hidden subgroup problem
`https://arxiv.org/abs/1112.3333`

OAYT Onuki, Aikawa, Yamazaki, and Takagi:
A Faster Constant-time Algorithm of CSIDH keeping Two Torsion Points
`https://eprint.iacr.org/2019/353.pdf`

Reg Regev:
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`https://arxiv.org/abs/quant-ph/0406151`

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Further reading:

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`https://arxiv.org/pdf/1806.03656`

DPV Decru, Panny, and Vercauteren:
Faster SeaSign signatures through improved rejection sampling
`https://eprint.iacr.org/2018/1109`

JLLR Jao, LeGrow, Leonardi, Ruiz-Lopez:
A polynomial quantum space attack on CRS and CSIDH
(MathCrypt 2018)

MR Meyer, Reith:
A faster way to the CSIDH
`https://ia.cr/2018/782`

Credits: thanks to Lorenz Panny for many of these slides, including all of the beautiful tikz pictures.