Non-interactive post-quantum key-exchange from isogeny graphs of elliptic curves

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[ˈsiːˌsaɪd]

History

1976 Diffie-Hellman: Key exchange using exponentiation in groups (DH)

1985 Koblitz-Miller: Diffie-Hellman style key exchange using multiplication in elliptic curve groups (ECDH)

1990 Brassard-Yung: Generalizes 'group exponentiation' to 'groups acting on sets' in a crypto context

1994 Shor: Polynomial-time quantum algorithm to break the discrete logarithm problem in any group, quantumly breaking DH and ECDH

1997 Couveignes: Post-quantum isogeny-based Diffie-Hellman-style key exchange using commutative group actions (not published at the time)

2003 Kuperberg: Subexponential-time quantum algorithm to attack cryptosystems based on a hidden shift

History

2004 Stolbunov-Rostovtsev independently rediscover Couveignes' scheme (CRS)

2006 Charles-Goren-Lauter: Build hash function from supersingular isogeny graph

2010 Childs-Jao-Soukharev: Apply Kuperberg's (and Regev's) hidden shift subexponential quantum algorithm to CRS

2011 Jao-De Feo: Build Diffie-Hellman style key exchange from supersingular isogeny graph (SIDH)

2018 De Feo-Kieffer-Smith: Apply new ideas to speed up CRS

2018 Castryck-Lange-Martindale-Panny-Renes: Apply ideas of De Feo, Kieffer, Smith to supersingular curves over $\mathbb{F}_p$ (CSIDH)

(History slides mostly stolen from Wouter Castryck)

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- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim 85\,\text{ms}$ for a full key exchange
- Flexible:
  - Compatible with 0-RTT protocols such as QUIC
  - [DG] uses CSIDH for ‘SeaSign’ signatures
  - [DGOPS] uses CSIDH for oblivious transfer
  - [FTY] uses CSIDH for authenticated group key exchange

Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group G via the map

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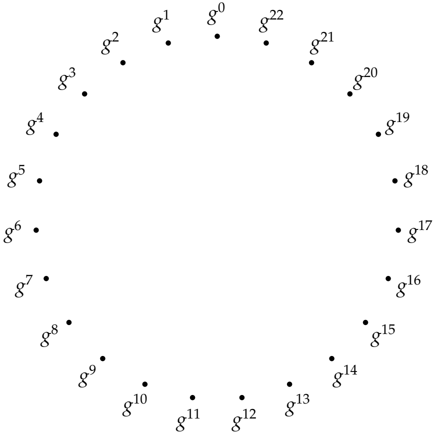
$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Square-and-multiply

Suppose $G \cong \mathbb{Z}/23$ and that Alice computes g^{13}.



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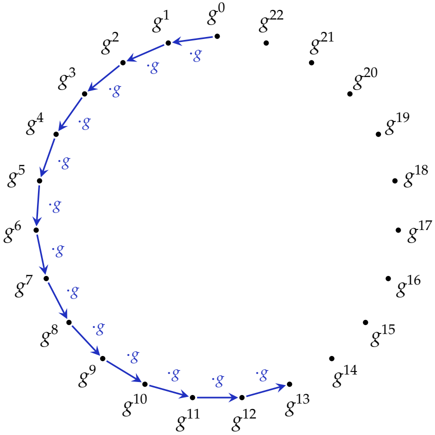

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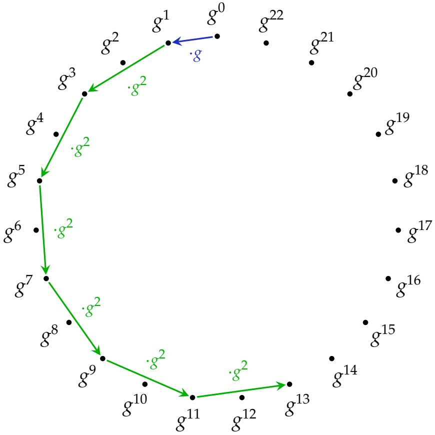

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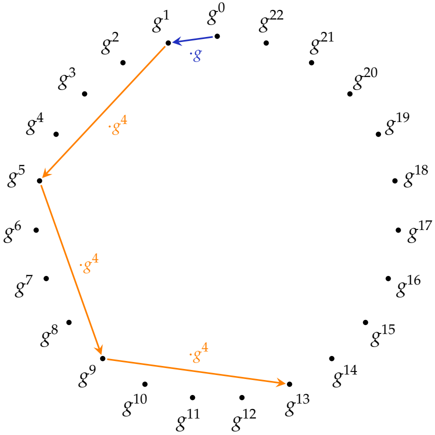

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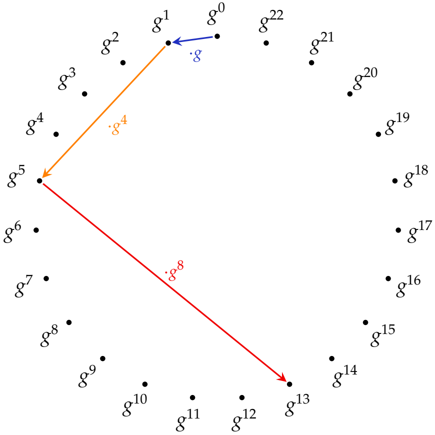

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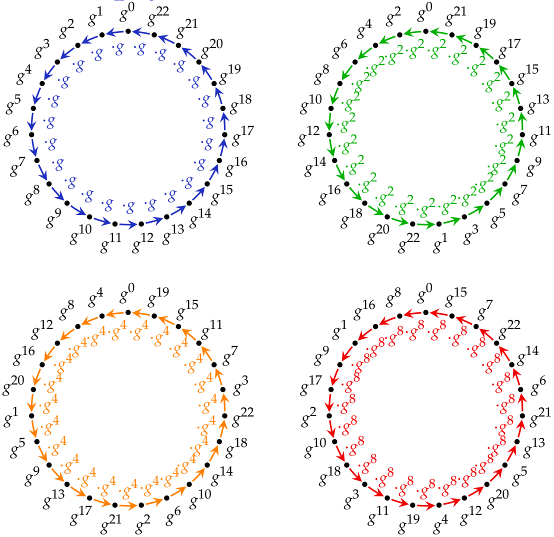

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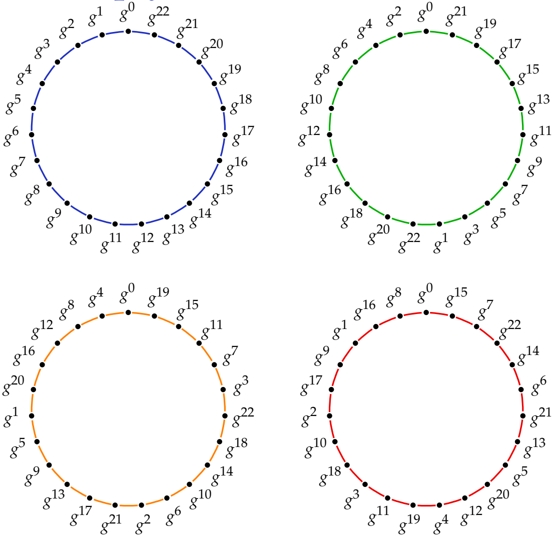

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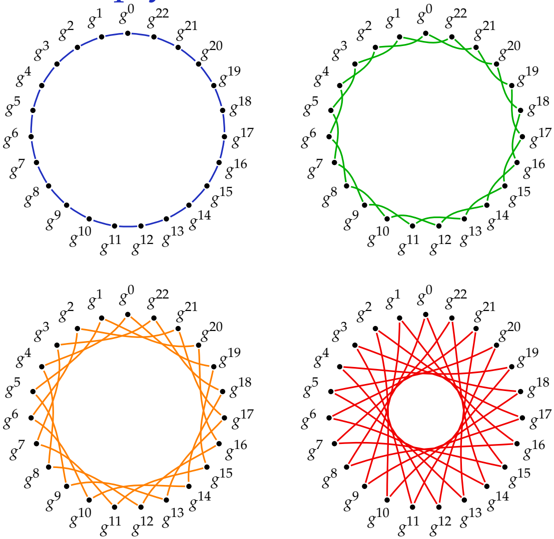

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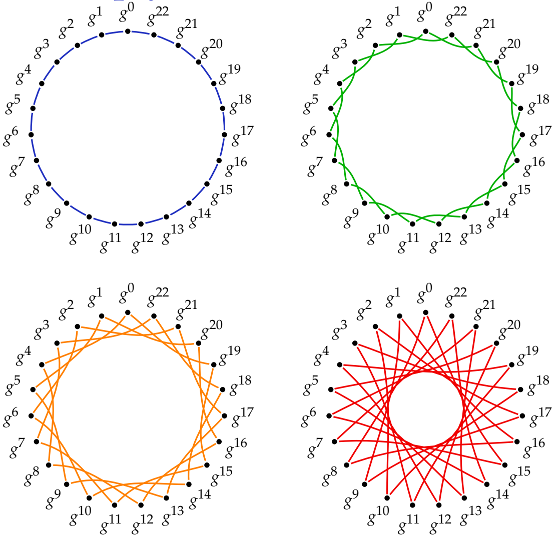
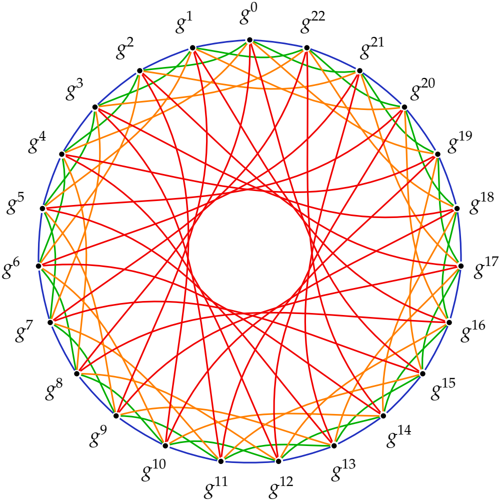


Cycles are compatible: [right, then left] = [left, then right], etc.

Union of cycles: rapid mixing



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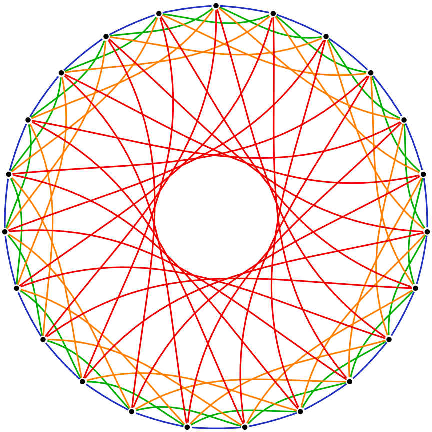
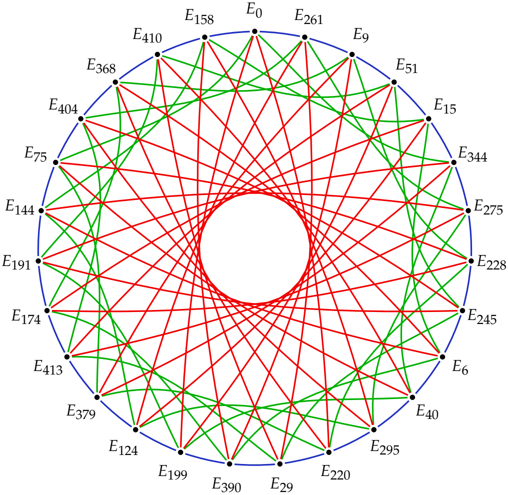


CSIDH: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



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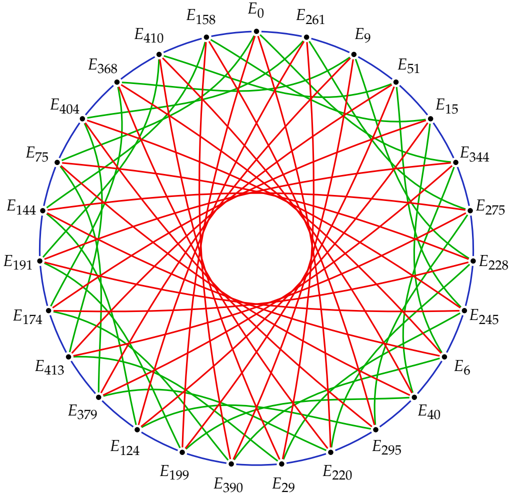

Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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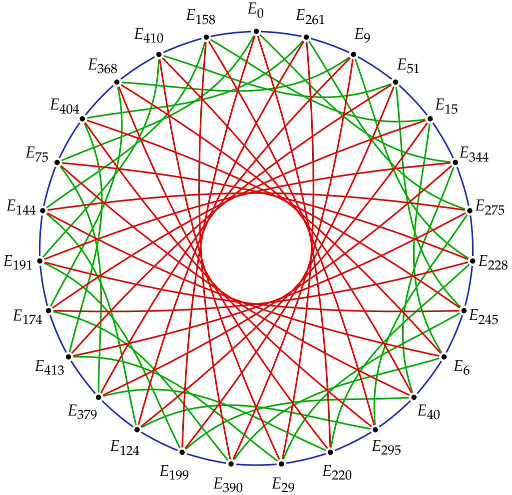

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Edges: 3-, 5-, and 7-isogenies.

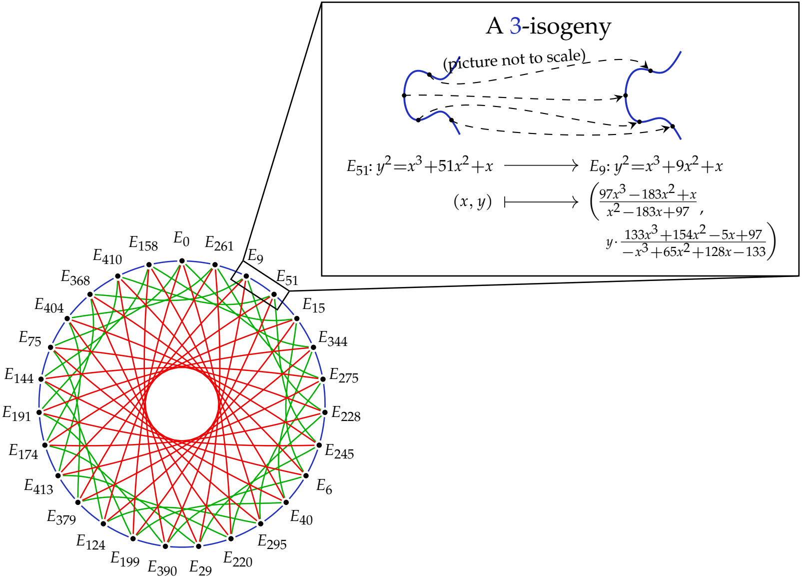
Quantumifying Exponentiation

- We want to replace the exponentiation map

$$\begin{array}{ccc} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x \end{array}$$

  by a group action on a set.
- Replace G by the set S of supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
- Replace $\mathbb{Z}$ by a commutative group H... more details to come!
- The action of a well-chosen $h \in H$ on S moves the elliptic curves one step around one of the cycles.

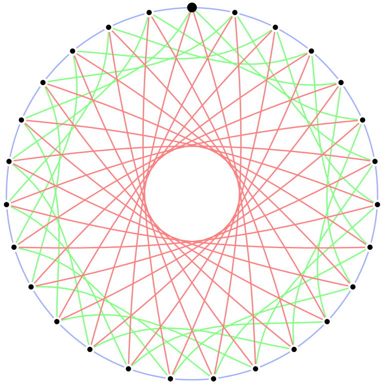
Graphs of elliptic curves



Diffie-Hellman on ‘nice’ graphs

Alice

[+, −, +, −]



Bob

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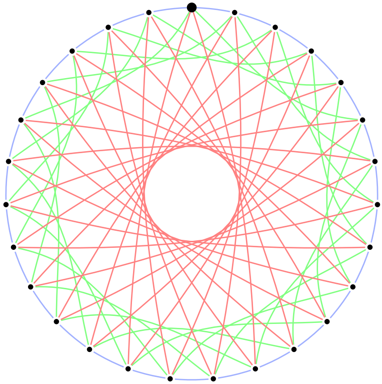

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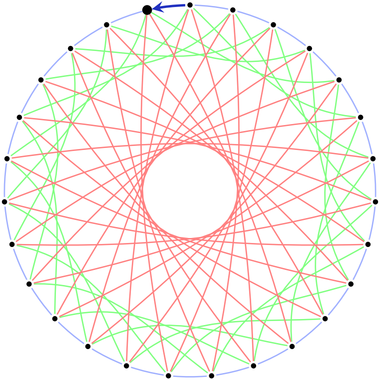

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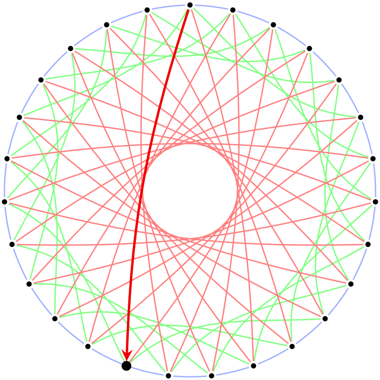

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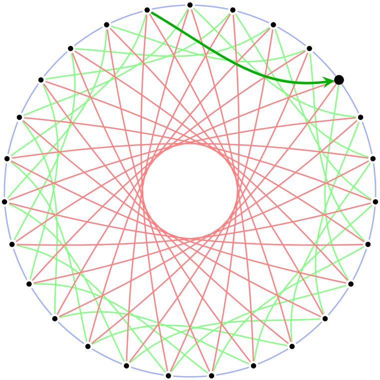

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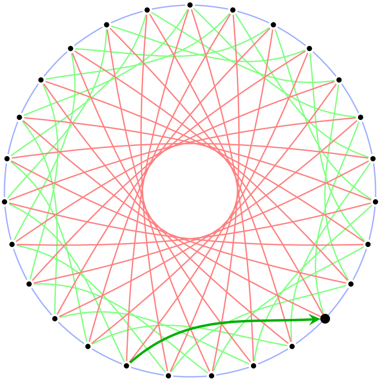

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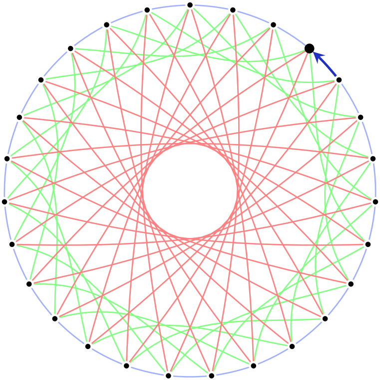

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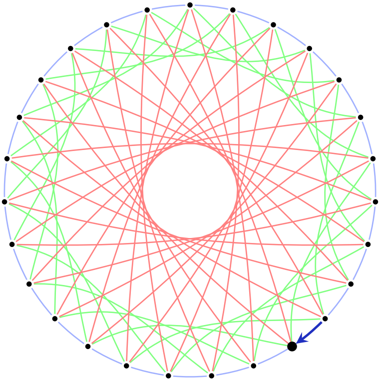

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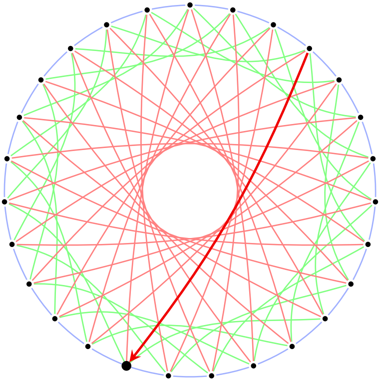

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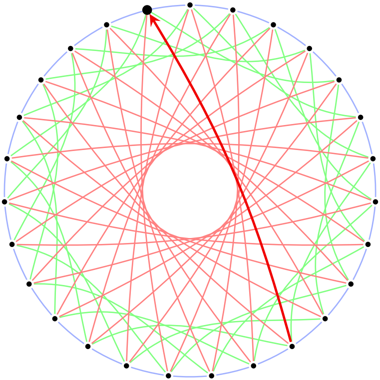

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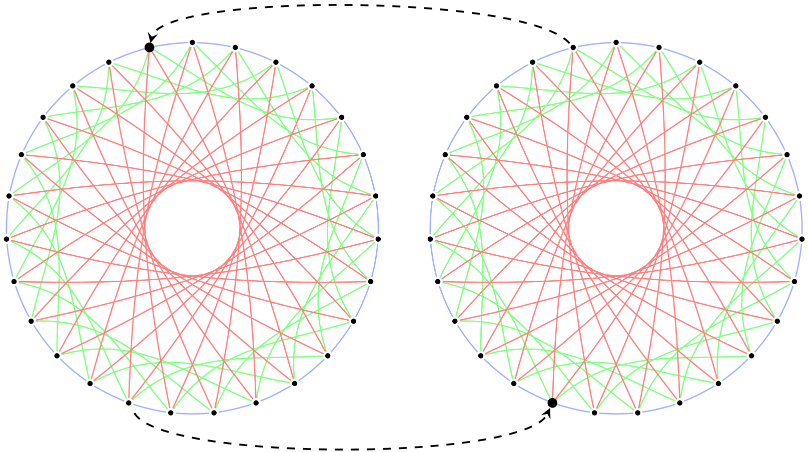

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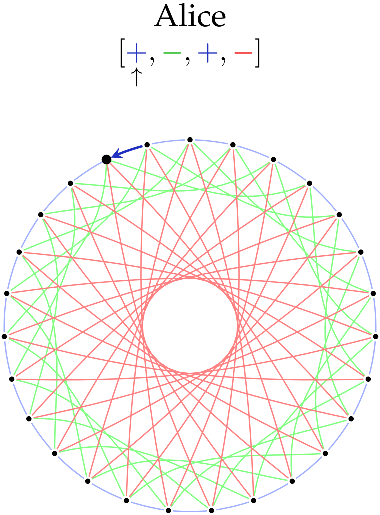

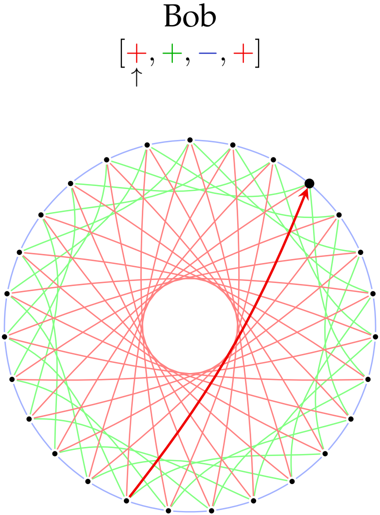

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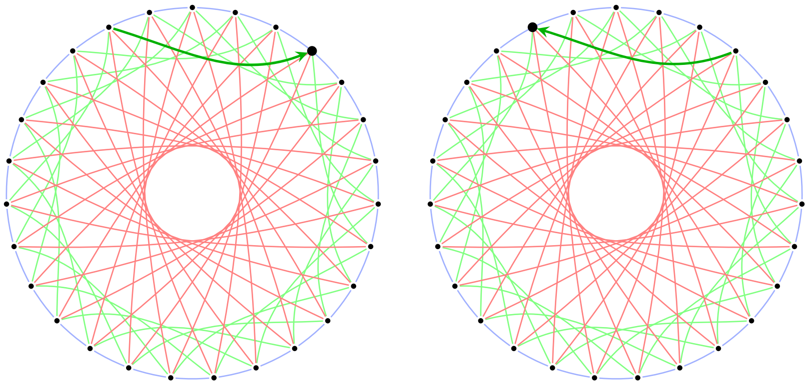

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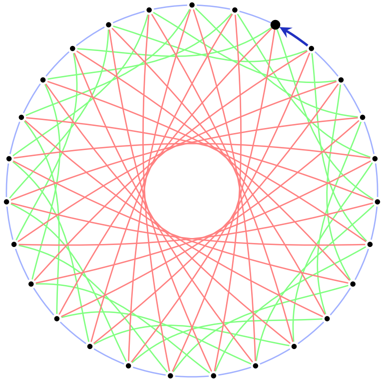

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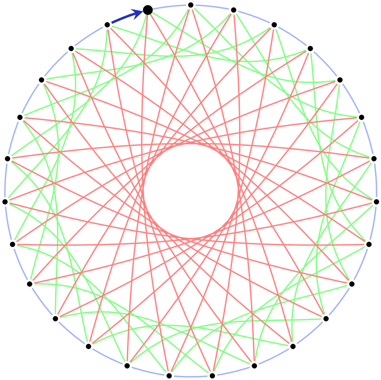

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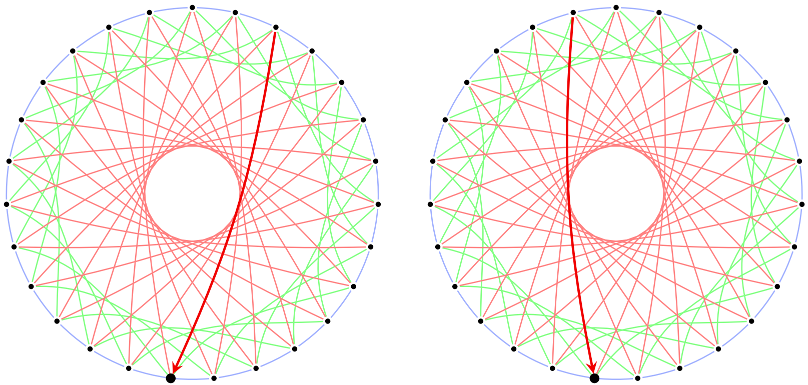

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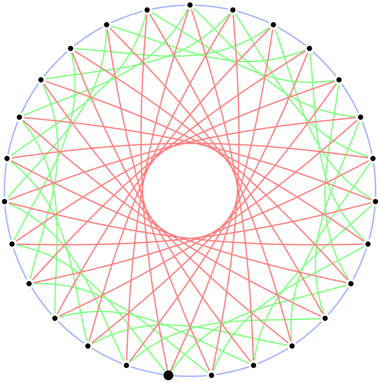

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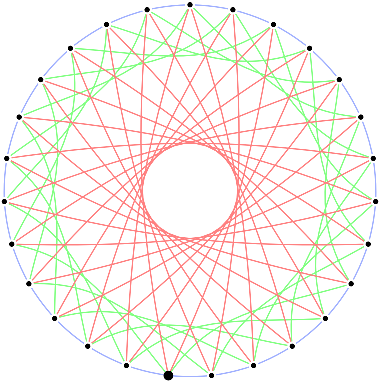

A walkable graph

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Important properties for such a walk:

IP1 ▸ The graph is a composition of compatible cycles.

IP2 ▸ We can compute neighbours in given directions.

Towards IP1: Isogeny graphs

First some reminders:

- An elliptic curve $E/\mathbb{F}_p$ (for $p \geq 5$) is supersingular if $\#E(\mathbb{F}_p) = p + 1$.

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- If $f : E \to E'$ is an ℓ-isogeny, there is a unique dual isogeny $f^\vee : E' \to E$ such that $f^\vee \circ f = [\ell]$ is the multiplication-by-ℓ map on E.

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- The dual isogeny is also an ℓ-isogeny.

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Let p and ℓ be distinct primes. The isogeny graph G_ℓ containing $E/\mathbb{F}_p$ is the graph with:

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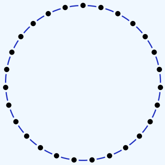
- In our example, these are

G_3:



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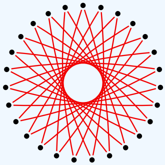

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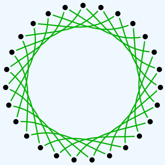

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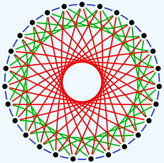
$G_3 \cup G_5 \cup G_7$:



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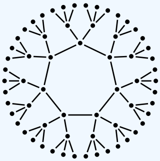
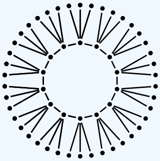
- Generally, the G_ℓ look something like

G_3: 

G_5: 

Towards IP1: Endomorphism rings

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- Two nodes are at different distances from the cycle if and only if they have different endomorphism rings.

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An endomorphism of an elliptic curve E is a morphism $E \to E$ (as abelian varieties).

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- The Frobenius map

$$\begin{array}{rccc} \pi: & E & \to & E \\ & (x,y) & \mapsto & (x^p, y^p) \end{array}$$

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Example

Let $p > 3$, let $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ be a supersingular elliptic curve, and let π be the Frobenius endomorphism. Then

$$\pi \circ \pi = [-p]$$

and

$$\begin{array}{ccc} \mathbb{Z}[\sqrt{-p}] & \to & \mathrm{End}_{\mathbb{F}_p}(E) \\ n & \mapsto & [n] \\ \sqrt{-p} & \mapsto & \pi \end{array}$$

extends $\mathbb{Z}$-linearly to a ring homomorphism.

Towards IP1: Group action

For $p \equiv 3 \pmod 8$ and $p \geq 5$, if $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ is supersingular, then $\mathrm{End}_{\mathbb{F}_p}(E_A) \cong \mathbb{Z}[\sqrt{-p}]$.

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- What is the action?

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- For $[I] \in \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}])$, let $\tilde{I}$ be an integral representative of the ideal class $[I]$. Then
$$\begin{array}{ccc} \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}]) \times S & \to & S \\ ([I], E) & \mapsto & f_{H_{\tilde{I}}}(E) \end{array}$$
is a free, transitive group action!

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$\rightsquigarrow$ there is a choice of $\ell_1, \ldots, \ell_n$ such that $G_{\ell_1} \cup \cdots \cup G_{\ell_n}$ is a composition of compatible cycles (IP1).

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- Choosing the direction in the graph corresponds to choosing this sign.

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- Given a $\mathbb{F}_p$-rational point of order ℓ, the isogeny computations can be done over $\mathbb{F}_p$.

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- If $p \equiv -1 \pmod{\ell}$ then the action $[\langle \ell, \pi + 1 \rangle] * E$ gives an ℓ-isogeny in the '$-$' direction.

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For which ℓ can we (efficiently) compute the neighbours of supersingular $E/\mathbb{F}_p$ in its ℓ-isogeny graph G_ℓ for odd $\ell|(p+1)$?

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Given the group action as above, Vélu's formulas give actual isogenies!
With our design choices all isogeny computations are over $\mathbb{F}_p$. [1]

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$\Rightarrow$ Tiny keys!

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p + 1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p + 1]P = \infty$.[2]

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- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from E_0 to E_A, whereas an attacker has compute all the possible paths from E_0.
- Best classical attacks are (variants of) meet-in-the-middle: Time $O(\sqrt[4]{p})$.

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- Kuperberg later [Kup2] gave more trade-off options for quantum and classical memory vs. time.
- Childs-Jao-Soukharev [CJS] applied Kuperberg/Regev to CRS – their attack also applies to CSIDH.
- Part of CJS attack computes many paths in superposition.

Quantum Security

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
  - Choice of time/memory trade-off (Regev/Kuperberg)
  - Quantum evaluation of isogenies

(and much more).

[3] From [BLMP], using query count of [BS]. [BS] also study quantum evaluation of isogenies but their current preprint misses some costs.

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  - Quantum evaluation of isogenies

  (and much more).
- Most previous analysis focussed on asymptotics

[3] From [BLMP], using query count of [BS]. [BS] also study quantum evaluation of isogenies but their current preprint misses some costs.

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- For fastest variant of Kuperberg (uses billions of qubits), total cost of CSIDH-512 attack is about 2^{81} qubit operations.[3]

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Work in progress & future work

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- Hardware implementation.
- More applications.
- Exploit more types of isogeny graphs (e.g. of abelian surfaces).
- [Your paper here!]


Thank you!

References

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Kup1 Kuperberg:
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JLLR Jao, LeGrow, Leonardi, Ruiz-Lopez:
A polynomial quantum space attack on CRS and CSIDH
(MathCrypt 2018)

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