From Conic Sections to Isogeny Graphs

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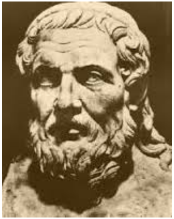
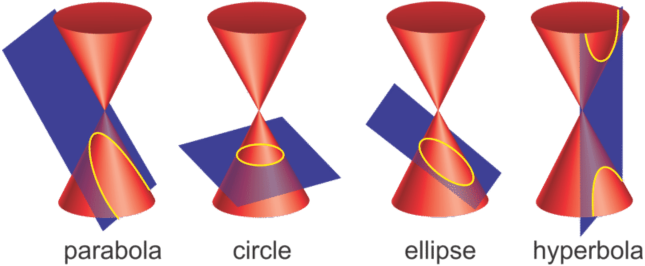
Diophantine equations through the ages

- **c. 360-350 BC: Menachmus**
- c. 350 BC: Euclid
- c. 250 AD: Diophantus writes *Arithmetica*





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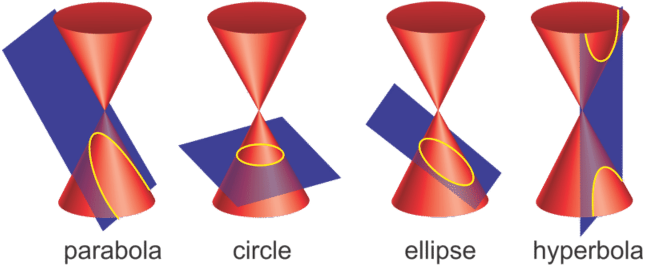

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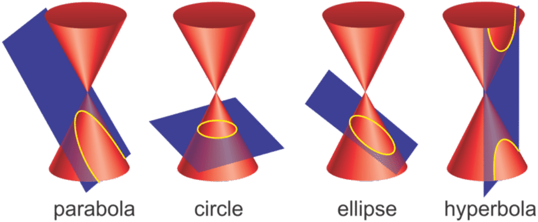
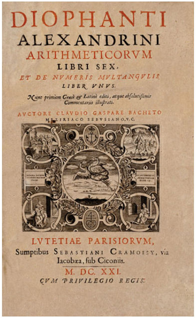


$$ax^2 + by^2 = 1$$


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Diophantine equations through the ages: two variables

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where $f \in \mathbb{Z}[x, y]$.

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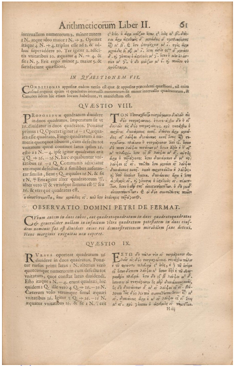
Pierre de Fermat (1601-1665)

- Showed that all algebraic curves over $\mathbb{Q}$ of degree 2 are conics
- Claimed to have a proof that there are no non-trivial rational solutions to the algebraic curve $x^n + y^n = 1$ for $n \geq 3$.



Arithmeticorum Liber II. 61

IN QVÆSTIONEM VII.

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

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Isaac Newton (1643-1727)

- 1710: classifies algebraic curves over $\mathbb{Q}$ of degree 3, showing by that by applying rational transformations to x and y, these curves can always be written as

$$y^2 = x^3 + ax + b$$

for some $a, b \in \mathbb{Z}$.

Examples of algebraic curves of degree 3

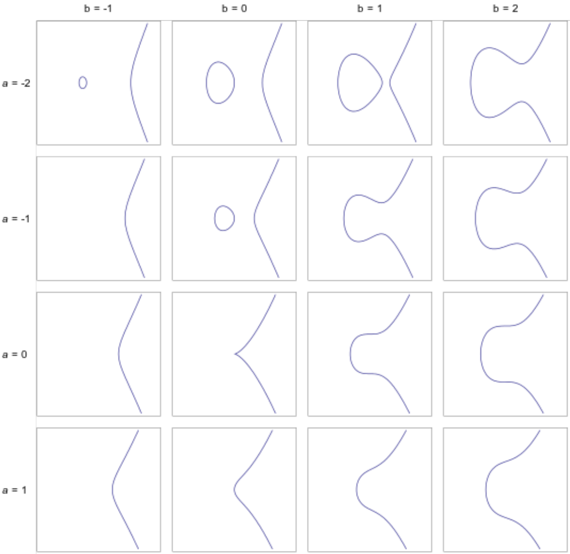

Image credits: "EllipticCurveCatalog" by Tos - Own work. Licensed under Public Domain via Commons

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Interlude: Algebraic integers

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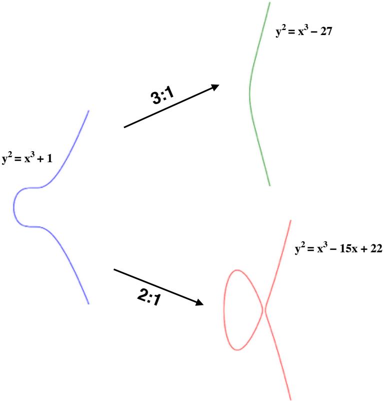
Two μ-isogenies of algebraic curves of degree 3



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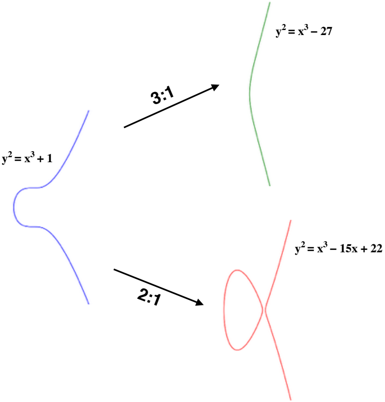

These maps are explicit, for example the top map is:

$$(x, y) \mapsto \left(\frac{x^3 + 4}{x^2}, \frac{x^3 y - 8y}{x^3} \right).$$

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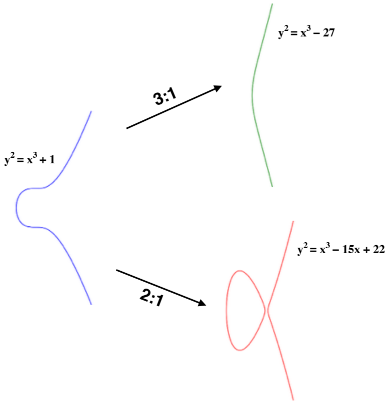
- edges given by μ-isogenies.

Playing with algebraic curves

Example



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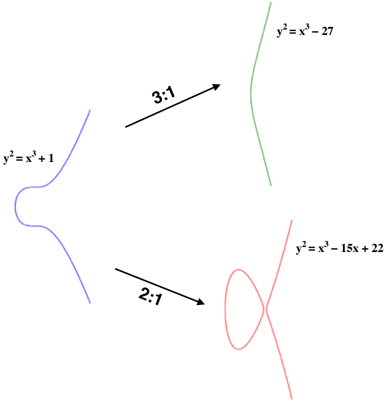

We can draw a 3-isogeny graph of the top isogeny:

And a 2-isogeny graph of the bottom isogeny:

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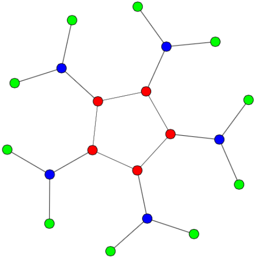

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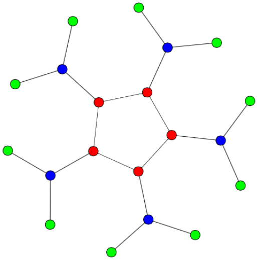

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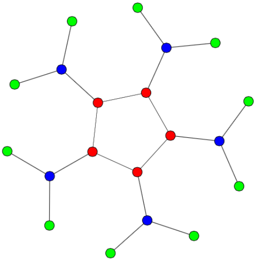

- contains a unique cycle, and
- has exactly v edges from every vertex, except for the vertices joined to the cycle by a path of exactly d edges, from which there is exactly 1 edge.

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Theorem (Kohel 1996)

If $\mu \in \mathbb{Z}$, then the connected components of G are volcano graphs.

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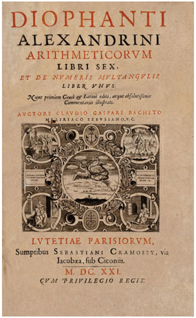
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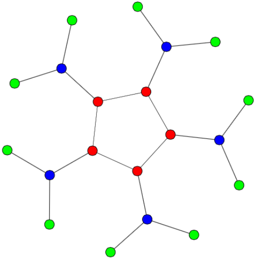

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- Explicit descent via μ-isogeny