Quantum attacks on isogeny-based cryptography

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Based on joint work with
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quantum.isogeny.org


[ˈsiːˌsaɪd]

Why CSIDH?

- Drop-in post-quantum replacement for (EC)DH
- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Smallest keys of all post-quantum key exchange candidates
- Competitive speed: 50-60ms for a full key exchange

Post-quantum Diffie-Hellman?

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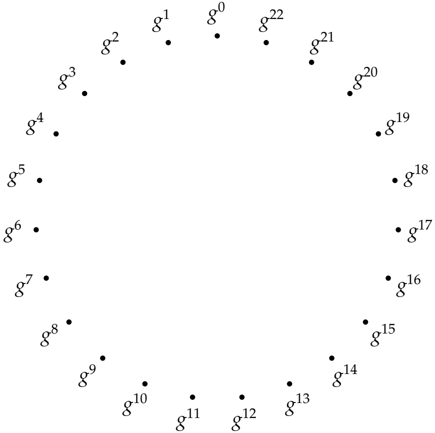
Square-and-multiply

Suppose $G \cong \mathbb{Z}/23$ and that Alice computes g^{13}.



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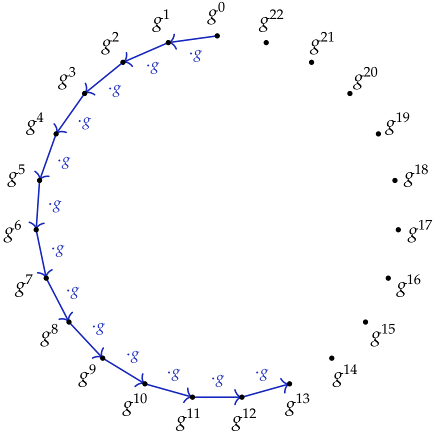

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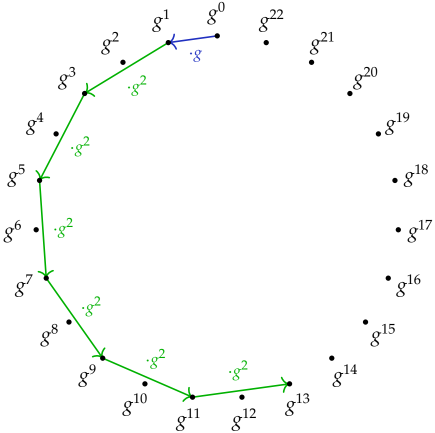

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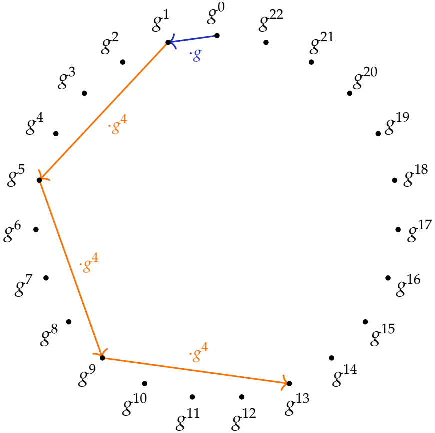

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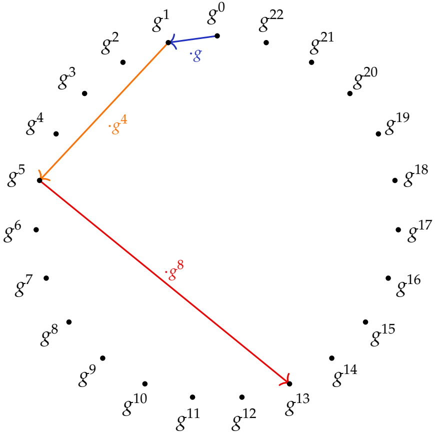

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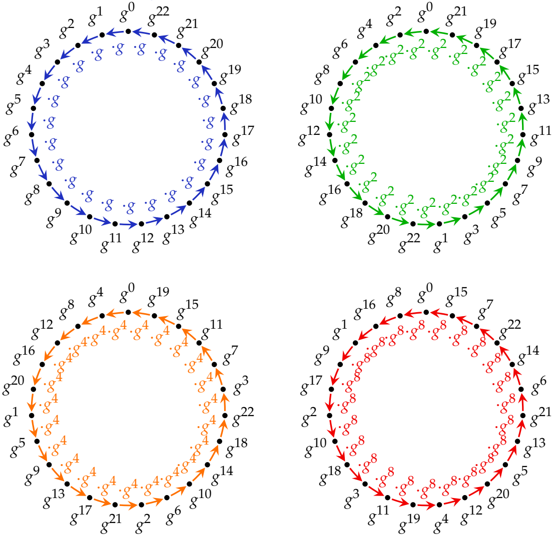

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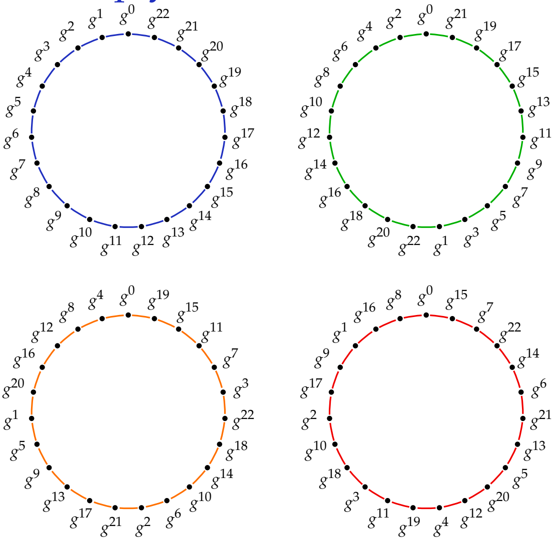

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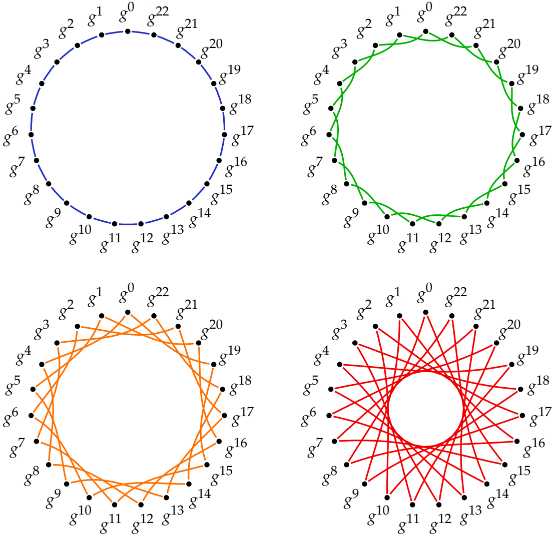

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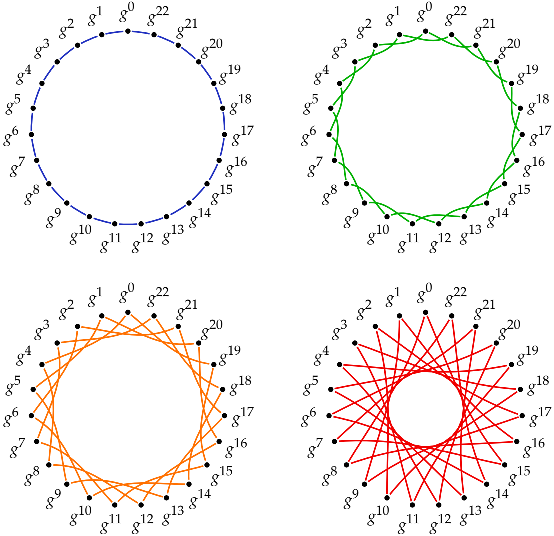
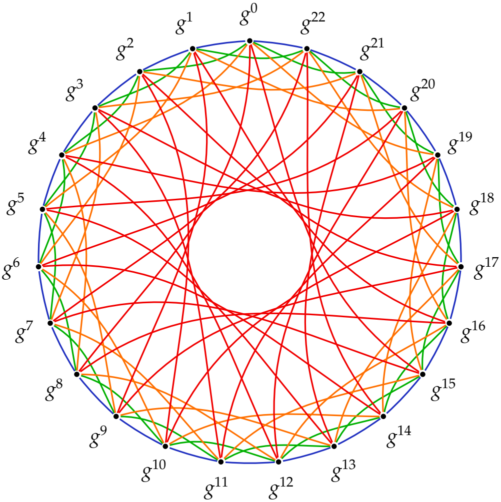


Cycles are compatible: [right, then left] = [left, then right], etc.

Union of cycles: rapid mixing



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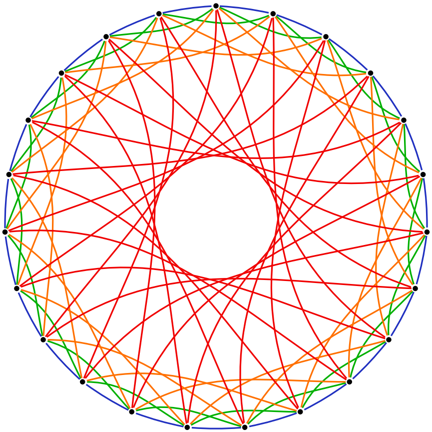
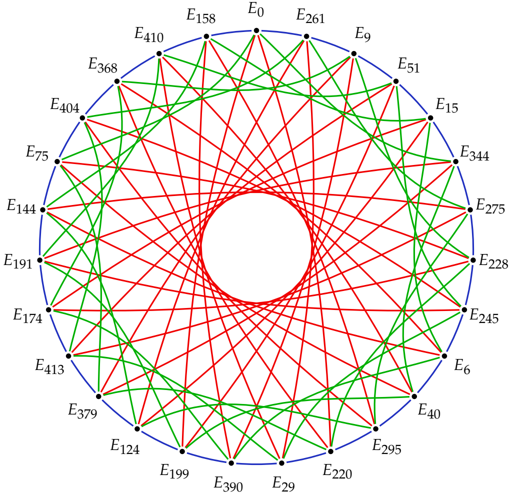


CSIDH: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



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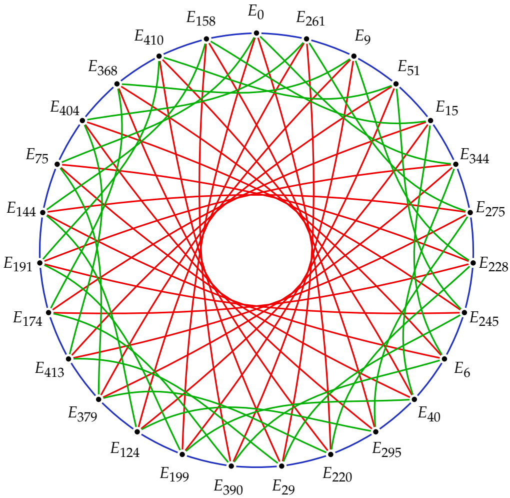

Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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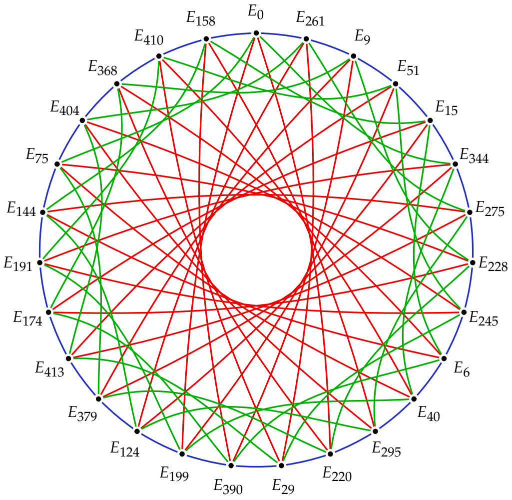

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Edges: 3-, 5-, and 7-isogenies.

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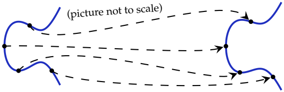
  - Vélu's formulas:

$$\text{generators of kernel} \rightsquigarrow \text{rational maps}$$

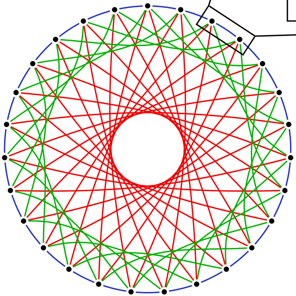
Graphs of elliptic curves

A 3-isogeny



$$E_{51}\colon y^2 = x^3 + 51x^2 + x \longrightarrow E_9\colon y^2 = x^3 + 9x^2 + x$$

$$(x, y) \longmapsto \left(\frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97}, \; y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$$



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- We want to replace the exponentiation map

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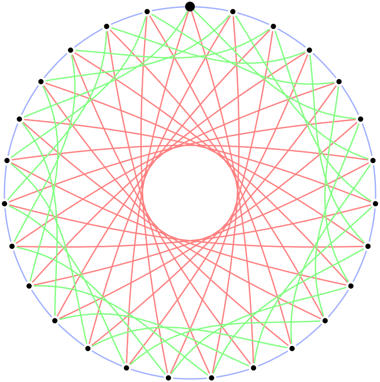
*Die-hards: $H = \mathrm{cl}(\mathrm{End}_{\mathbb{F}_p}(E)) = \mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$; an ideal class $[I] \in H$ defines the kernel.

Diffie-Hellman on ‘nice’ graphs

Alice

$[+, -, +, -]$



Bob

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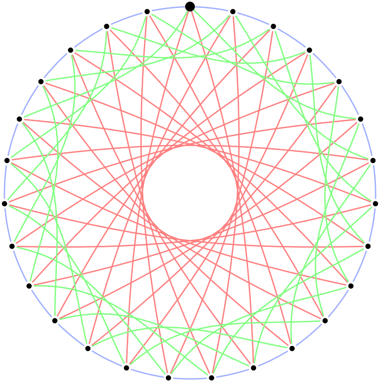

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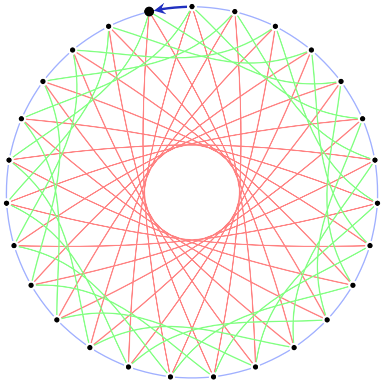

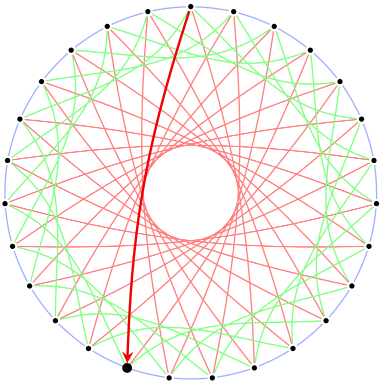

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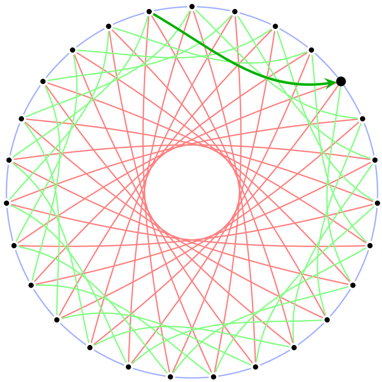

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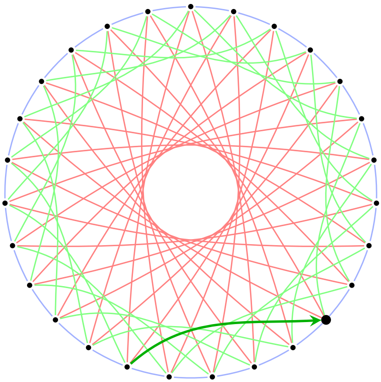

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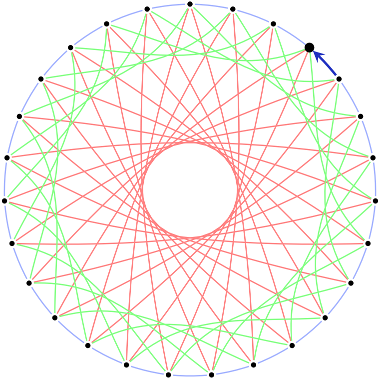

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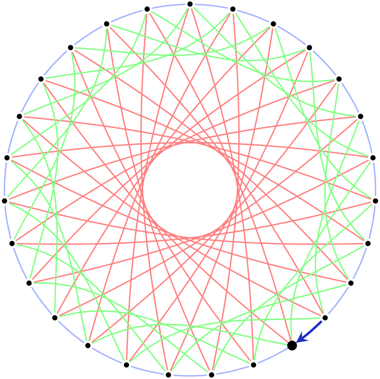

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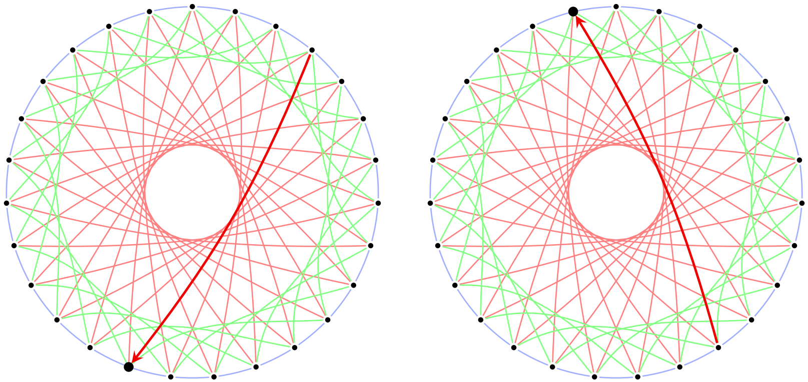

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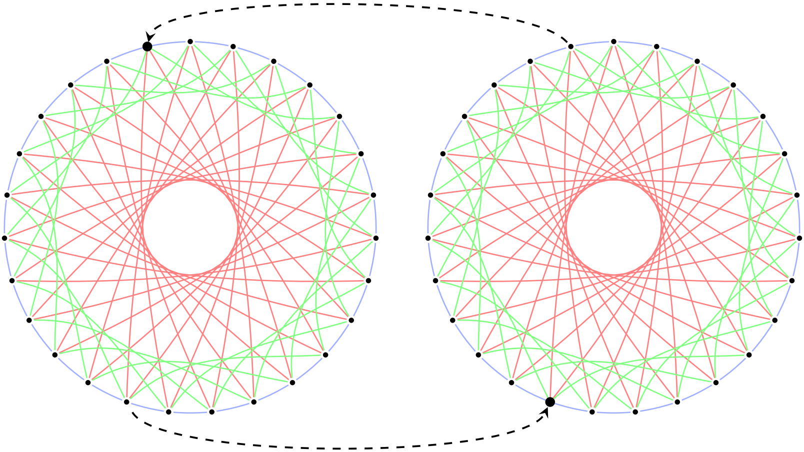

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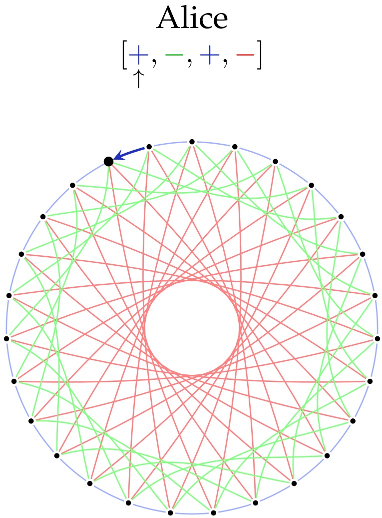

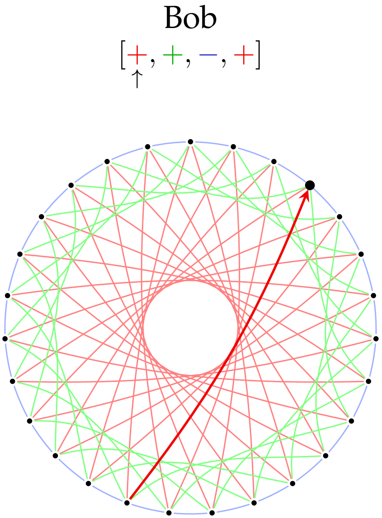

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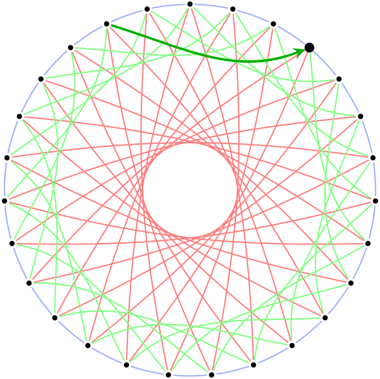

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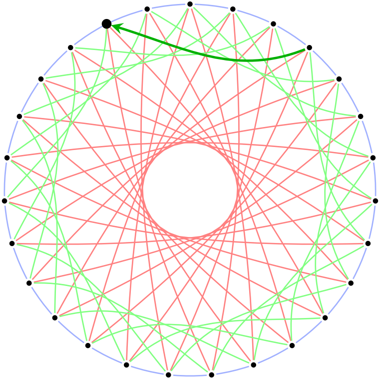

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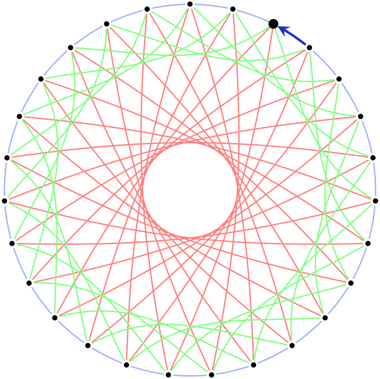

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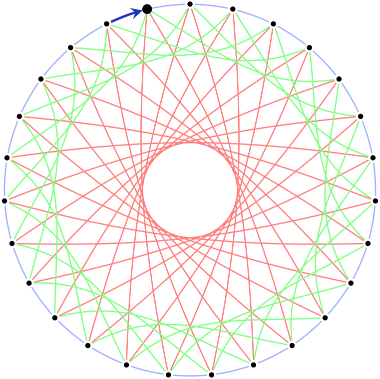

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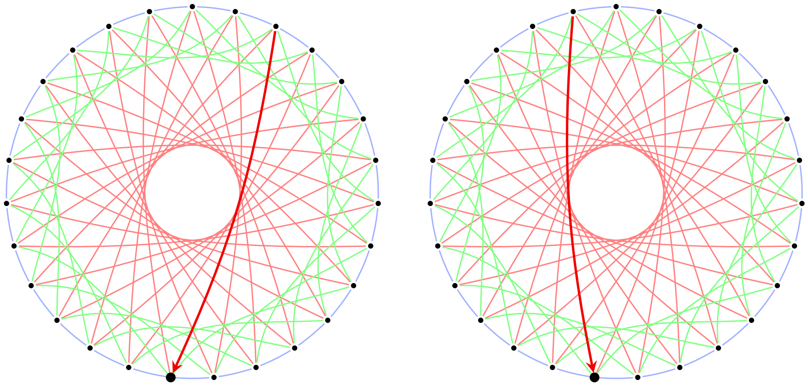

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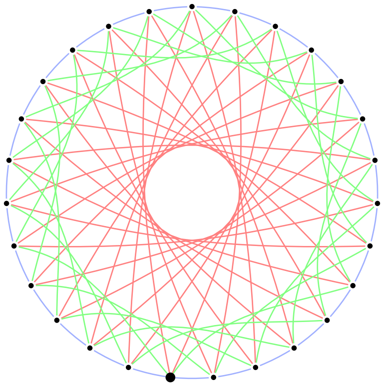

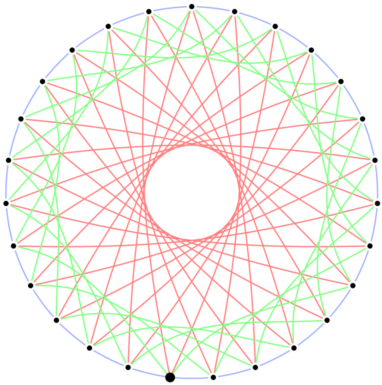

Quantum security: from CSIDH to DHSP

- Hard problem in CSIDH: given group action

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  and $s_0, s_1 \in S$, find $\chi \in H$ such that $\chi \cdot s_0 = s_1$.

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- $\varphi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ a homomorphism, and
- $$\begin{array}{rcl} (\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}) & \to & \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z} \\ (a_1, b_1), (a_2, b_2) & \mapsto & (a_1\varphi(b_1)(a_2), b_1 b_2). \end{array}$$

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$$\begin{aligned} f(h,a) = f(h',a') \Leftrightarrow & \, a = 0,\, a' = 1,\, h' = h\chi, \text{ or} \\ & \, a = 1,\, a' = 0,\, h = h'\chi, \text{ or} \\ & \, a = a' = 1,\, h = h'. \end{aligned}$$

$\rightsquigarrow$ f hides the subgroup $\{(1,0),(\chi,1)\} \subset H \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$.

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$\rightsquigarrow f$ hides the subgroup $\{(1,0),(\chi,1)\} \subset H \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$.
- Finding subgroup hidden by f gives secret χ.

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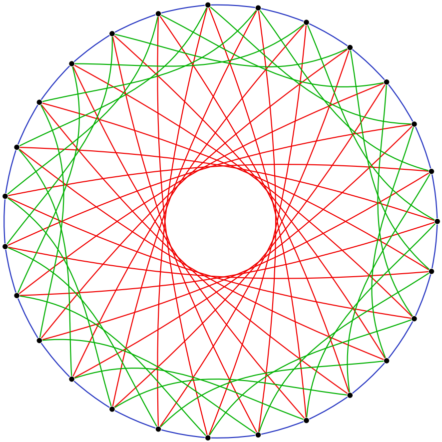
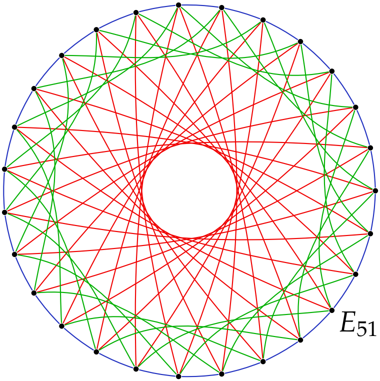
How expensive is each CSIDH query?



Secret key: path on the graph
Public key: end points of path
One query: computes many paths in superposition

Computing isogenies

Aim: given curve E_A, find a neighbour in the isogeny graph



Edges: 3-, 5-, and 7-isogenies.

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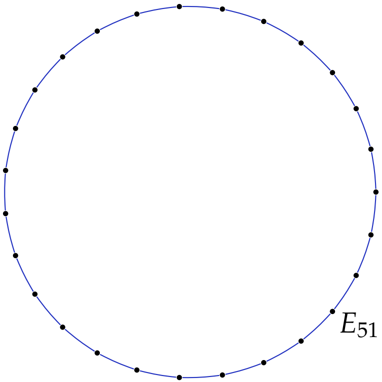

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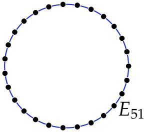

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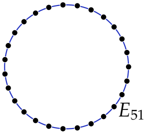

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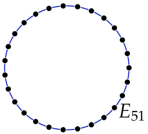

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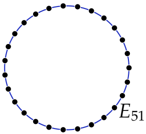

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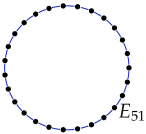

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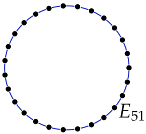

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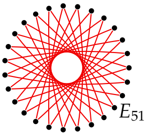

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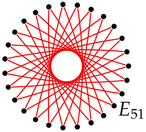

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- Choose a random $\mathbb{F}_{419}$-point $P = (x, y)$ on E_{51}
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- With probability $\frac{4}{5}$, $84 \cdot P$ has order 5
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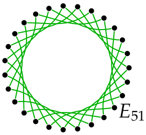
Computing isogenies

Aim: given curve E_A, find a neighbour in the 7-isogeny graph



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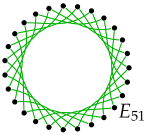

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- With probability $\frac{6}{7}$, $60 \cdot P$ has order 7
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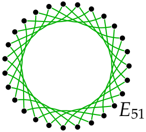
Computing isogenies

Aim: given curve E_A, find a neighbour in the ℓ-isogeny graph



- Recall: $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$
- Choose a random $\mathbb{F}_p$-point $P = (x, y)$ on E_A
- P has order dividing $p + 1$.
- With probability $\frac{\ell-1}{\ell}$, $\frac{p+1}{\ell} \cdot P$ has order ℓ.*
- Using Vélu's formulas, find map with kernel = $\langle \frac{p+1}{\ell} \cdot P \rangle$
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[BLMP] Gives many optimizations / more complex variants–trying to mitigate these problems.

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Open question:
How much faster than the generic conversion is possible?

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- Number of queries: $\approx 2^{19.3}$ using $\approx 2^{32}$ bits of QRACM [P].

Oracle errors

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- Understanding the error tolerance of Kuperberg's algorithm is essential to obtain accurate concrete numbers.
- Advances in quantum error correction would also massively change the complexity.

Open questions: summary

- How do oracle errors interact with Kuperberg's algorithm?
- What kind of overheads come from handling large numbers of qubits?
- Is there a quantum algorithm that does better than L(1/2)?
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Thank you!

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Credits to my coauthors Daniel J. Bernstein, Tanja Lange, and Lorenz Panny for many of the contents of this presentation.