CSIDH: An Efficient Post-Quantum Commutative Group Action

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The Discrete Logarithm Problem

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  eg. WhatsApp messages; internet banking apps; sites using 'https'.
- What is the discrete logarithm problem?

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then G is a group with group operation $*$ given by multiplication.

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- The discrete logarithm problem (DLP): given $g \in G$ and $\underbrace{g * \cdots * g}_{n \text{ times}}$, find n.

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then G is a group with group operation $*$ given by multiplication.
DLP in $(\mathbb{Z}/23\mathbb{Z}) - \{0\}$: Given $g \bmod 23$ and $g^n \bmod 23$, find n.

The Discrete Logarithm Problem

The DLP is hard when, given $g \in G$:

- Given $n \in \mathbb{Z}$, computing $\underbrace{g * \cdots * g}_{n \text{ times}}$ is fast. (eg. Polynomial time).

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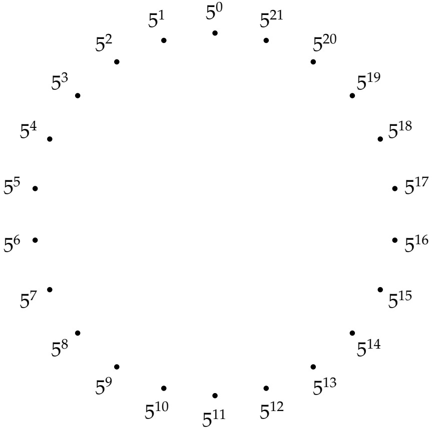
Example: Given $g = 5 \bmod 23$:

- Let $n = 9$; compute $5^9 \bmod 23$.
- If $5^n = 11 \bmod 23$; compute n.

Square-and-multiply

Compute 5^9 mod 23.



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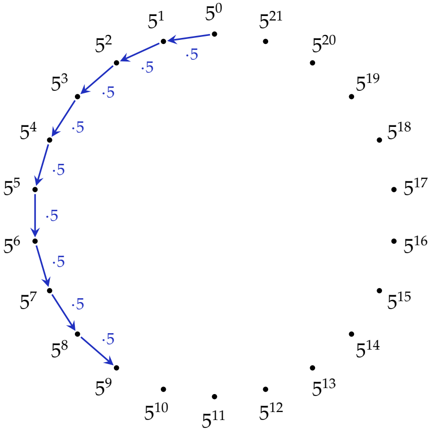

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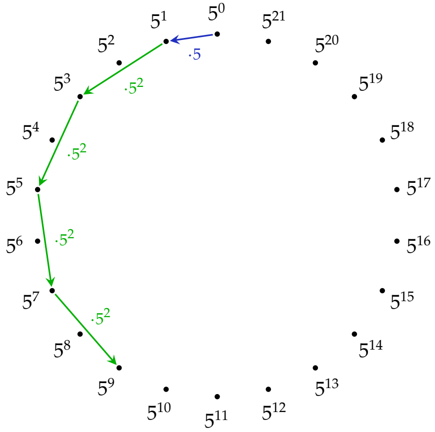

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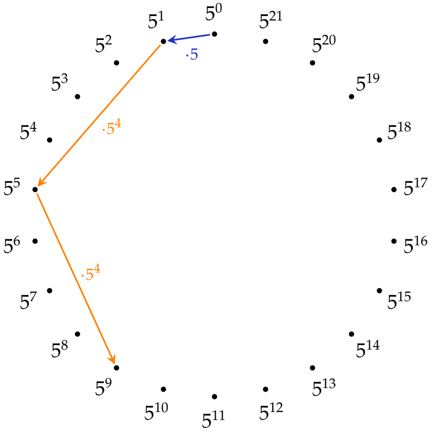

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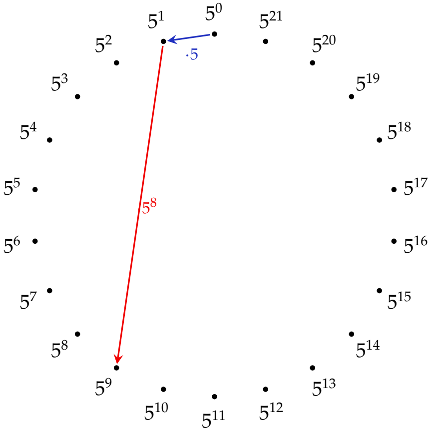

Square-and-multiply vs. solving DLP

- To compute $5^9 \bmod 23$, compute:
  $5 \cdot 5^8 = 5 \cdot ((5^2)^2)^2 \bmod 23$. (Fast).

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$$5^2 \equiv 2 \not\equiv 11 \bmod 23$$

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$$\begin{aligned}
5^2 &\equiv 2 \not\equiv 11 \bmod 23 \\
5^3 &\equiv 10 \not\equiv 11 \bmod 23 \\
5^4 &\equiv 4 \not\equiv 11 \bmod 23 \\
5^5 &\equiv 20 \not\equiv 11 \bmod 23 \\
5^6 &\equiv 8 \not\equiv 11 \bmod 23 \\
5^7 &\equiv 17 \not\equiv 11 \bmod 23 \\
5^8 &\equiv 16 \not\equiv 11 \bmod 23 \\
5^9 &\equiv 11 \bmod 23.
\end{aligned}$$

Square-and-multiply vs. solving DLP

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  $5 \cdot 5^8 = 5 \cdot ((5^2)^2)^2$ mod 23. (Fast).
- To compute n such that $5^n \equiv 11$ mod 23, check:

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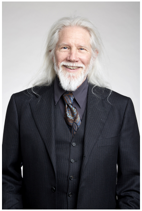
$$5^9 \equiv 11 \bmod 23.$$

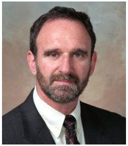
(Slow).

(There are smarter ways to do this in practise, but they're still slow).

Application of DLP: Diffie-Hellman key exchange



$g \in G$



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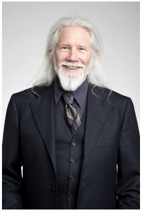

Secret key: d

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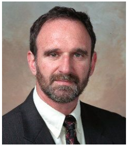

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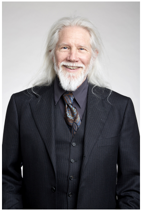

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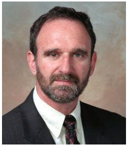
$g \in G$

$\xrightarrow{\text{Public key: } g^d}$

$\xleftarrow{\text{Public key: } g^h}$



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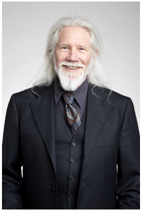

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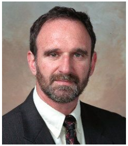

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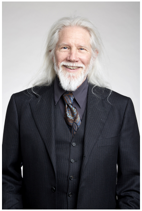

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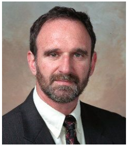

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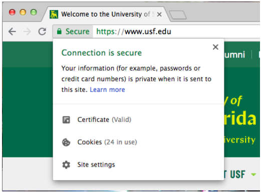
If DLP is hard for G, then computing the public keys and the shared secret is fast for Diffie and Hellman, and computing the secret values is slow for an adversary.

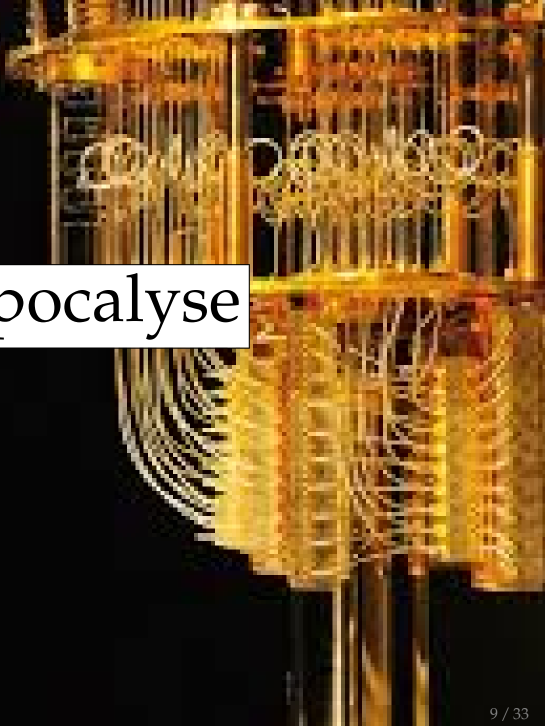
Applications of Diffie-Hellman key exchange

The Diffie-Hellman key exchange is a building block in:

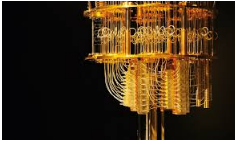
- Digital signature schemes (used for example by some online banking apps; secure websites).
- Encrypted messaging services (eg. WhatsApp).



Cryptapocalyse



Quantum cryptapocalyse

$g^n \longrightarrow$  $\longrightarrow n$

Shor's algorithm quantumly computes n from g^n and g in any group in polynomial time. (About as fast as computing g^n from n and g).

$\rightsquigarrow$ All applications of DLP are broken by quantum computers!


QUANTUM COMPUTING
~~RALPH~~
BREAKS THE
INTERNET
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Quantum cryptapocalyse

Key Finding 10: Even if a quantum computer that can decrypt current cryptographic ciphers is more than a decade off, the hazard of such a machine is high enough – and the time frame for transitioning to a new security protocol is sufficiently long and uncertain – that prioritization of the development, standardization, and deployment of post-quantum cryptography is critical for minimizing the chance of a potential security and privacy disaster.

Report by the US National Academy of Sciences, see

`http://www8.nationalacademies.org/onpinews/newsitem.aspx?RecordID=25196`

Reminder: applications of Diffie-Hellman key exchange

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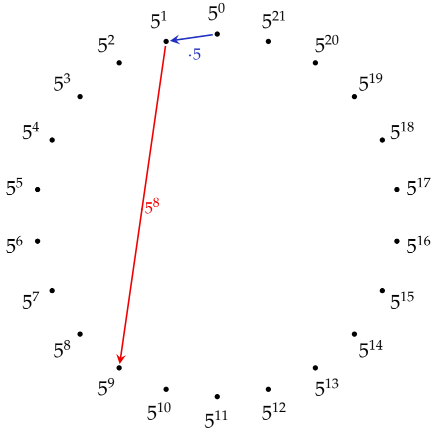
  - Digital signature schemes (used for example by some online banking apps; secure websites).
  - Encrypted messaging services (eg. WhatsApp).
- We need a post-quantum Diffie-Hellman-style key exchange.

Square-and-multiply

Reminder: how to compute 5^9 mod 23.



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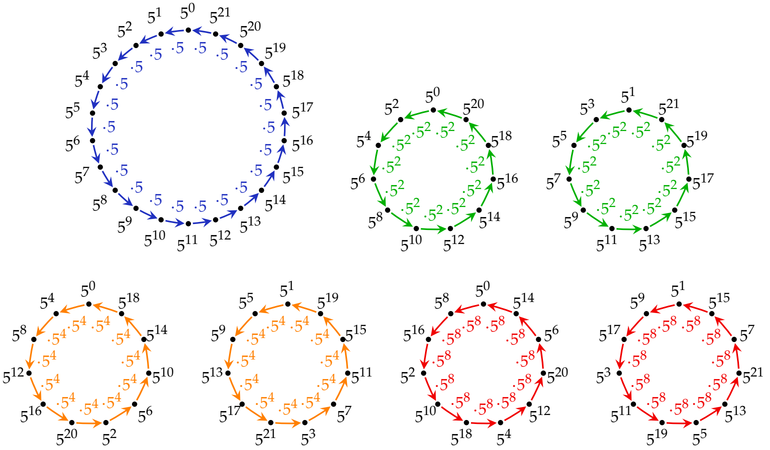

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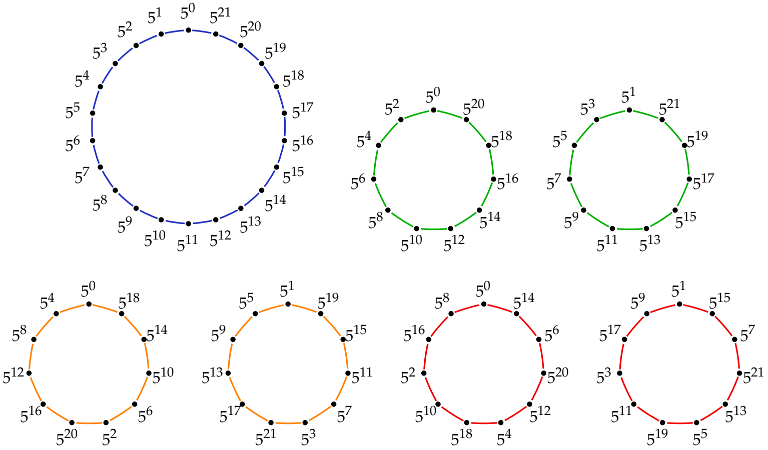

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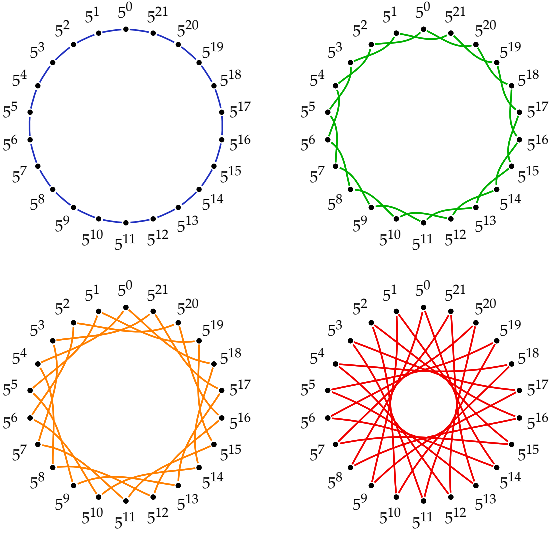

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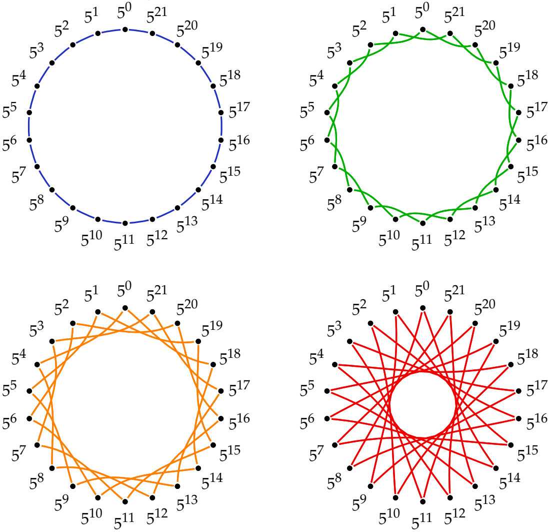
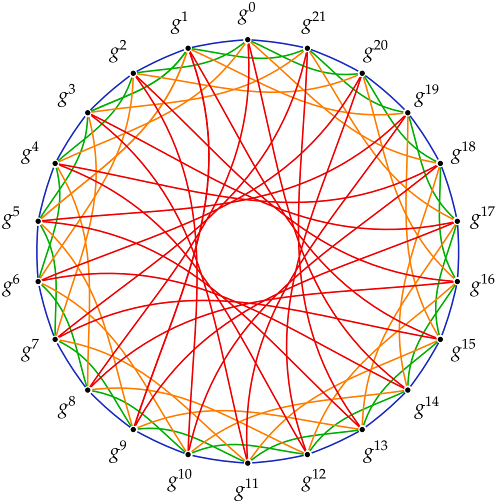


Needed for Diffie-Hellman: Cycles are compatible–
[right, then left] = [left, then right], etc. (Else $(5^a)^b \neq (5^b)^a$).

Union of cycles: rapid mixing



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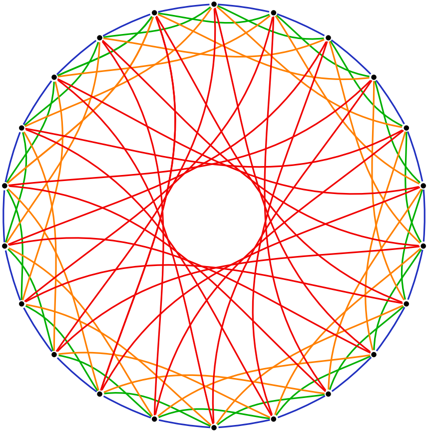
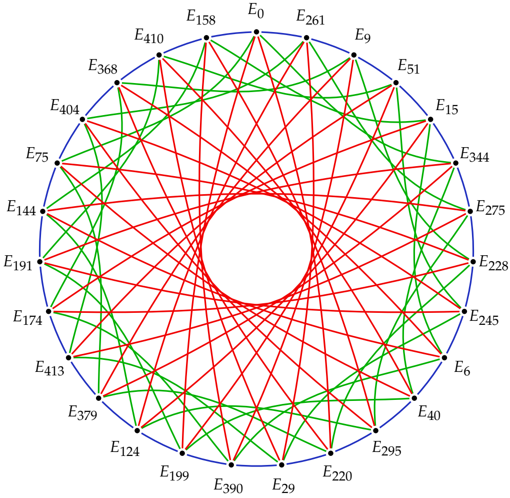


Post-quantum Diffie-Hellman: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



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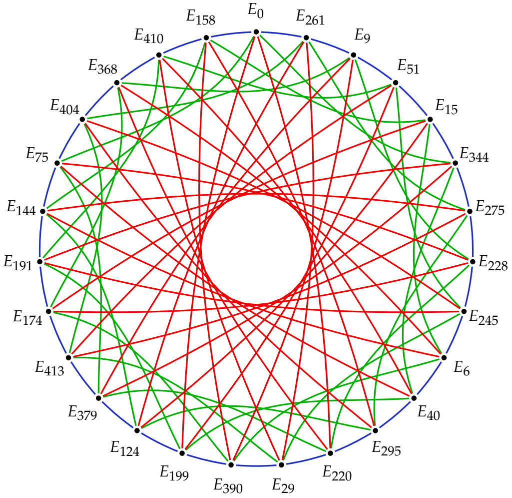

Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{Z}/419\mathbb{Z}$.

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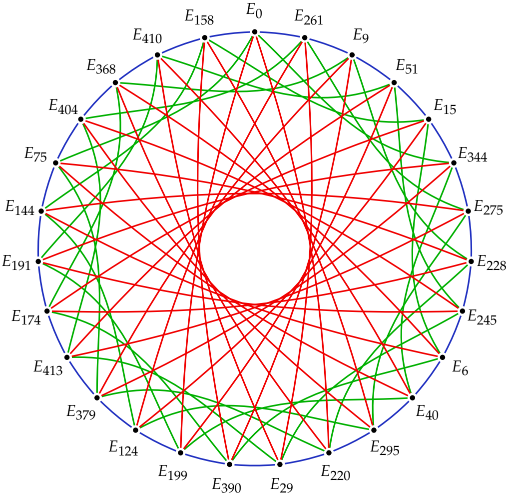

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Edges: 3-, 5-, and 7-isogenies.

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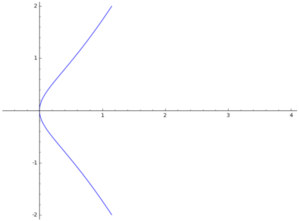

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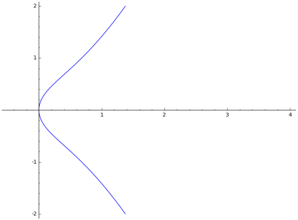
$A = 0$:



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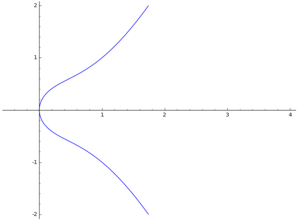
$A = -1$:



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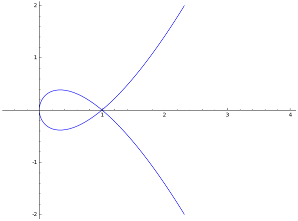

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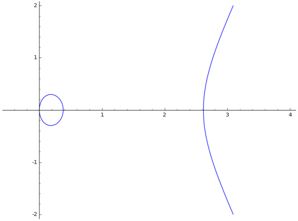

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Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

- If equation E_A is smooth (no self intersections or cusps) it represents an elliptic curve.
- The set of $\mathbb{F}_p$-rational solutions (x, y) to an elliptic curve equation $E_A/\mathbb{F}_p$, together with a 'point at infinity' P_∞, forms a group with identity P_∞, notated $E_A(\mathbb{F}_p)$.

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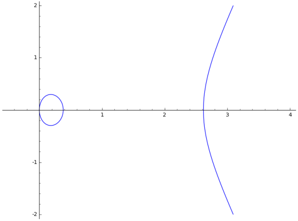

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- An elliptic curve $E_A/\mathbb{F}_p$ with $p \geq 5$ such that $\#E_A(\mathbb{F}_p) = p + 1$ is supersingular.

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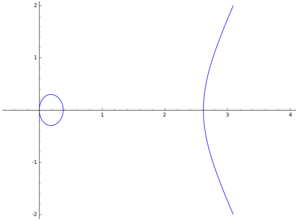

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Edges: 3-, 5-, and 7-isogenies.

- An isogeny $E_A \to E_B$ is a non-zero morphism the preserves P_∞ ('nice map' given by rational maps).

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- For $\ell \neq p$ (= 419 here), an ℓ-isogeny $f : E_A \to E_B$ is an isogeny with $\#\ker(f) = \ell$.

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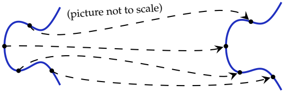
- For $\ell \neq p$ (= 419 here), an ℓ-isogeny $f : E_A \to E_B$ is an isogeny with $\#\ker(f) = \ell$.
- Every ℓ-isogeny $f : E_A \to E_B$ has a unique dual ℓ-isogeny $f : E_B \to E_A$.

Graphs of elliptic curves

A 3-isogeny



$$E_{51}\colon y^2=x^3+51x^2+x \longrightarrow E_9\colon y^2=x^3+9x^2+x$$

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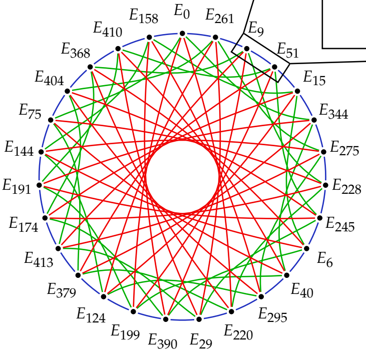
(...and its dual isogeny)



Diffie-Hellman on isogeny graphs

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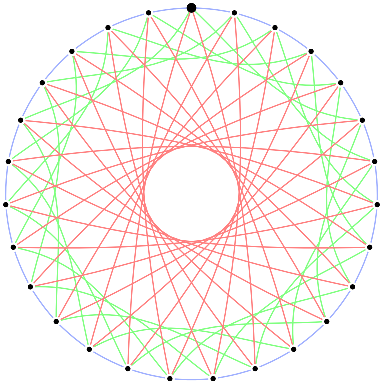

Bob

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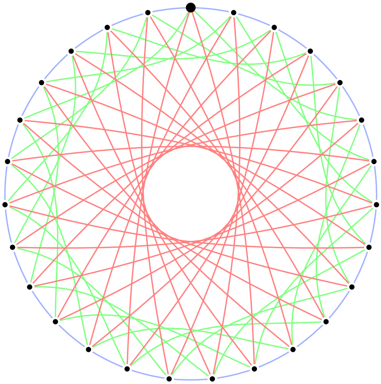

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$\uparrow$

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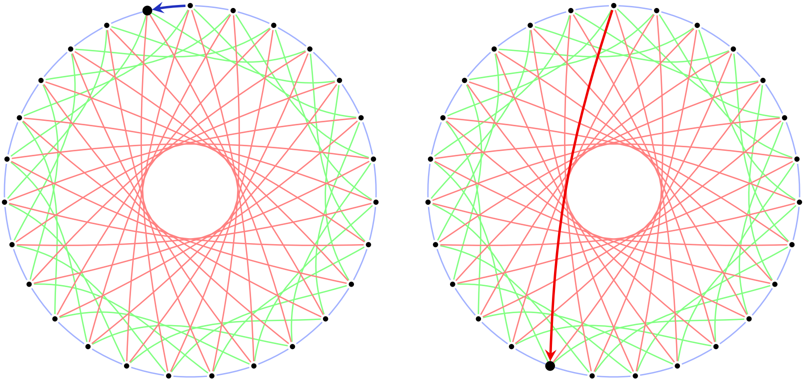

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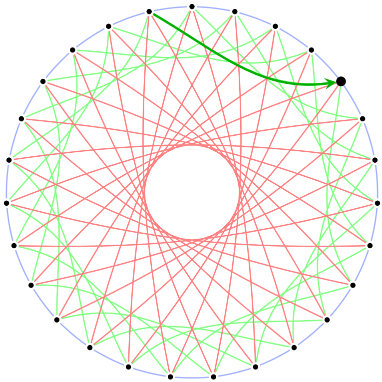

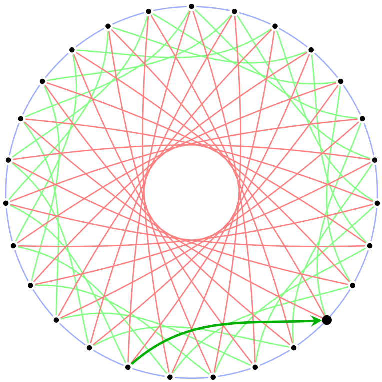

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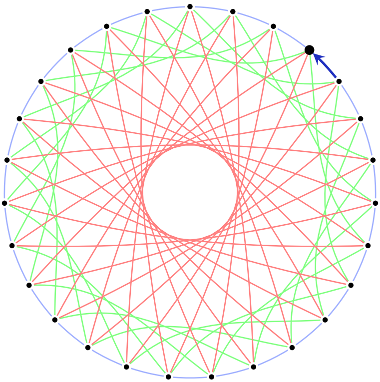

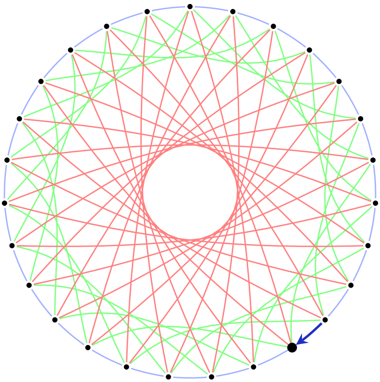

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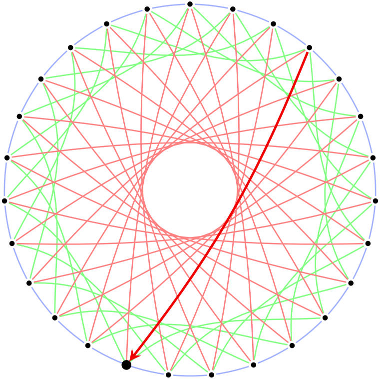

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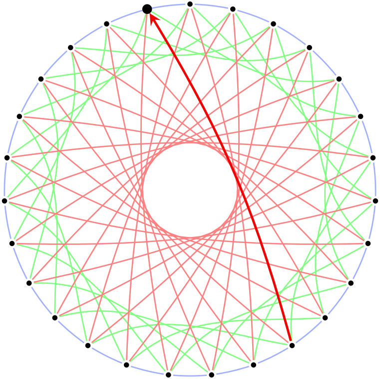

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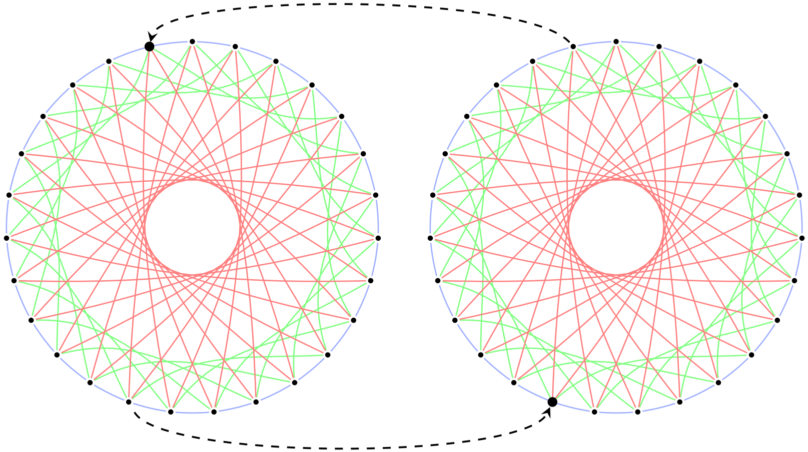

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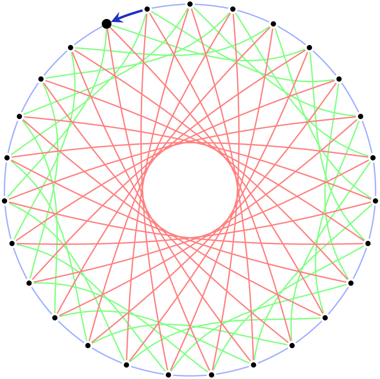

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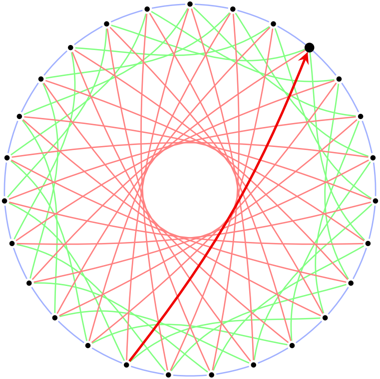

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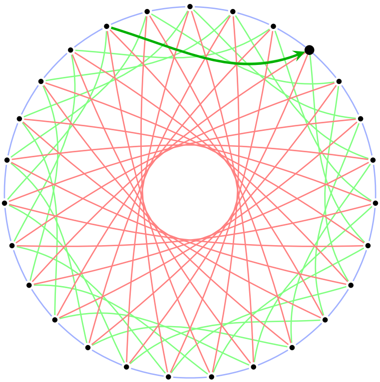

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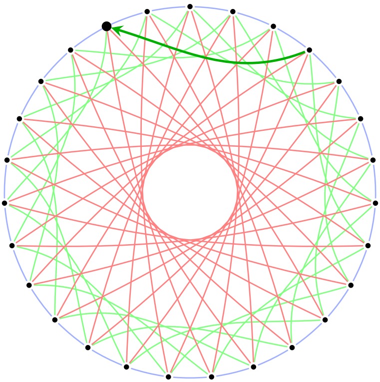

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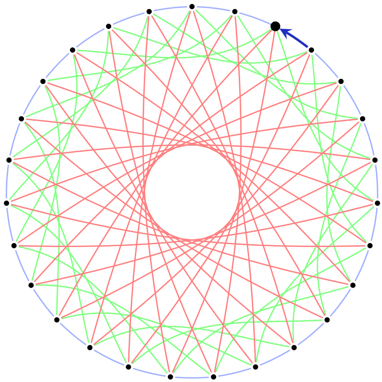

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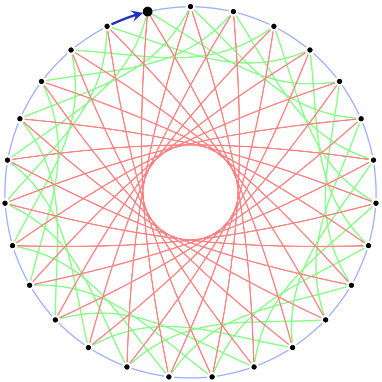

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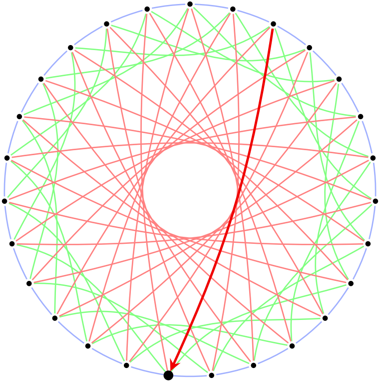

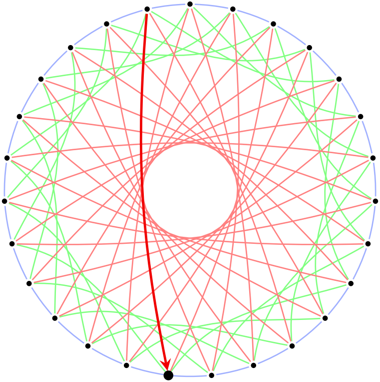

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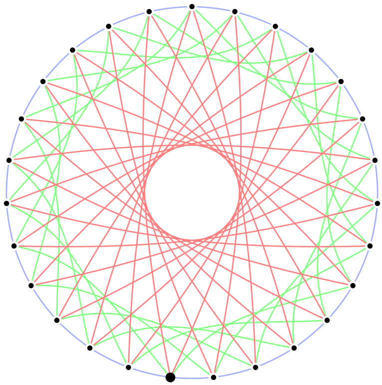

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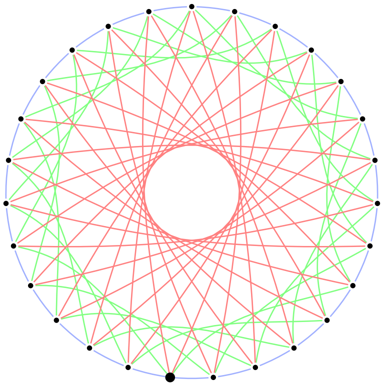

A walkable graph

Important properties for our graph:

- IP1 ▸ The graph is a composition of compatible cycles.
- IP2 ▸ We can compute neighbours in given directions.

IP1: A composition of cycles

- The graph used in CSIDH is constructed as a composition of graphs G_ℓ of ℓ-isogenies.

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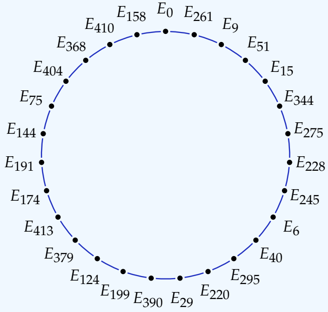

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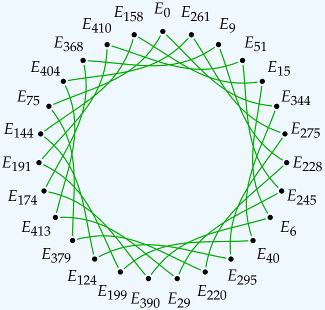

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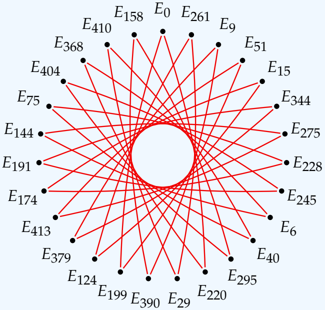

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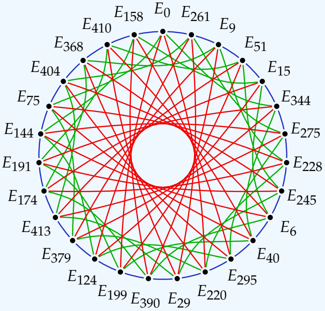
$G_3 \cup G_5 \cup G_7$:



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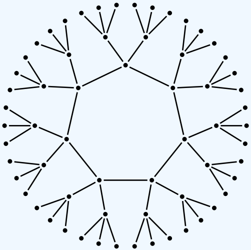

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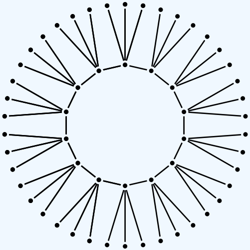

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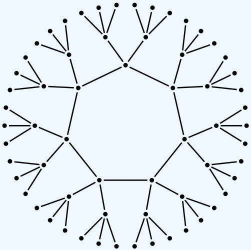
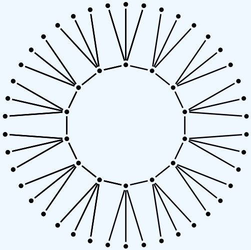
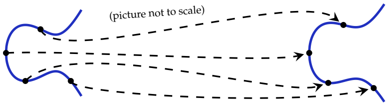
G_3:  G_5: 

- We want to make sure G_ℓ is just a cycle.

IP2: Compute neighbours in given directions

The edges of G_ℓ are ℓ-isogenies.



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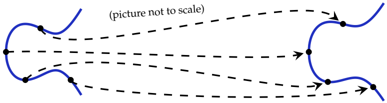

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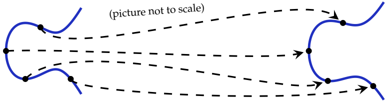

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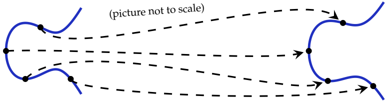

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- Generally needs big extension fields...

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   - All G_{ℓ_i} are compatible.
   - Computations need only $\mathbb{F}_p$-arithmetic (because $\ell_i | (p + 1)$).

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Side effect of magic:

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$\Rightarrow$ Tiny keys!

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p+1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p+1]P = \infty$.[1]

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- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from E_0 to E_A, whereas an attacker has compute all the possible paths from E_0.
- Best classical attacks are (variants of) meet-in-the-middle: Time $O(\sqrt[4]{p})$.

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- Part of CJS attack computes many paths in superposition.

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- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
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- Asymptotic complexity is relatively well understood [BIJ], [JLLR]
- [BLMP] gives full computer-verified simulation of quantum evaluation of isogenies $\rightsquigarrow$ concrete estimates for a given security level ('NIST level I')

Work in progress & future work

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- Explore different graph structures occuring for other curves/geometrical objects.
- More applications exploiting new graph structures.


Thank you!

Parameters

CSIDH-log p	intended NIST level	public key size	private key size	time (full exchange)	cycles (full exchange)	stack memory	classical security
CSIDH-512	1	64 b	32 b	70 ms	212e6	4368 b	128
CSIDH-1024	3	128 b	64 b				256
CSIDH-1792	5	224 b	112 b				448

CSIDH vs SIDH?

Apart from mathematical background, SIDH and CSIDH actually have very little in common, and are likely to be useful for different applications.

Here is a comparison for (conjectured) NIST level 1:

	CSIDH	SIDH
Speed (NIST 1)	65ms (can be improved)	$\approx$ 10ms[2]
Public key size (NIST 1)	64B	378B
Key compression (speed)		$\approx$ 15ms
Key compression (size)		222B
Constant-time slowdown	$\approx \times 2.2$ (can be improved)	$\approx \times 1$
Submitted to NIST	no	yes
Maturity	11 months	8 years
Best classical attack	$p^{1/4}$	$p^{1/4}$
Best quantum attack	$L_p[1/2]$	$p^{1/4}$
Key size scales	quadratically	linearly
Security assumption	isogeny walk problem	ad hoc
Non-interactive key exchange	yes	unbearably slow
Signatures (classical)	unbearably slow[3]	seconds
Signatures (quantum)	seconds	still seconds?

[2] This is a very conservative estimate!

[3] Word on the street is that a paper is coming with a signature scheme that takes milliseconds.

References

AMW Appelbaum, Martindale, and Wu:
Tiny Wireguard Tweak
(upcoming)

BLMP Bernstein, Lange, Martindale, and Panny:
Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies
`https://quantum.isogeny.org` (Eurocrypt 2019)

CLMPR Castryck, Lange, Martindale, Panny, Renes:
CSIDH: An Efficient Post-Quantum Commutative Group Action
`https://ia.cr/2018/383` (Asiacrypt 2018)

DG De Feo, Galbraith:
SeaSign: Compact isogeny signatures from class group actions
`https://ia.cr/2018/824`

DGOPS Delpech de Saint Guilhem, Orsini, Petit, and Smart:
Secure Oblivious Transfer from Semi-Commutative Masking
`https://ia.cr/2018/648`

FTY Fujioka, Takashima, and Yoneyama:
One-Round Authenticated Group Key Exchange from Isogenies
`https://eprint.iacr.org/2018/1033`